Scaling and Survival Properties of Random Walks with Absorbing and Moving Walls

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We study one-dimensional random walks between an absorbing boundary at the origin and a movable wall on the other end. The wall moves outward only, and only when the walker kicks it. The hopping of the walker has no directional bias, except at the location of the kicked wall. We propose a scaling ansatz for the probability distribution \( P(x,t) \) for a walker to be at \( x \) at time \( t \) as \( P(x,t) = t^{-\delta(q)/2} f(x/\langle x \rangle) \), where \( \delta(q) \) is a function of the bias hopping probability \( q \) on the wall and \( \langle x \rangle \) is the mean position of the walker. The scaling ansatz is numerically confirmed. From the numerical finding of \( \langle x \rangle = C t^{1/2} \) and the confirmed scaling relation, the survival probability of the walker can be shown to decay as \( t^{-\delta(q)} \) with a continuously varying critical exponent \( \delta(q) = 1/2q - 1/2 \). By setting up the relation between \( \langle x \rangle \) with the absorbing boundary and that without the absorbing boundary, we accurately estimate \( C \) as \( C = \sqrt{\pi/2} + \sqrt{2/\pi(2q - 1)/(1 - q)} \).

Nonuniversal dynamical behavior has appeared in various kinds of nonequilibrium models [1–8]. Typical examples are the systems that display an absorbing phase transition from an active phase into an absorbing phase with infinitely many absorbing states [1–4], random walks with a movable partial reflector [5], and directed self-avoiding walks confined to fixed parabolic geometries [6,7].

In systems exhibiting an absorbing phase transition with infinitely many absorbing states, the critical exponents that govern the spreading from a seed vary continuously with a particular density in the initial configuration. However, recently Grassberger et al. proposed that the spreading in systems with infinitely many absorbing configurations could be understood by studying a model with a unique absorbing state, but in which the spread of activity to virgin sites is controlled by a parameter [9]. They claimed that such a dynamic process gives rise to the nonuniversal behavior in models with infinitely many absorbing states.

As a simple version with the same idea, Dickman and ben-Avraham introduced a one-dimensional lattice random walk with an absorbing boundary at the origin and a movable partial reflector at the maximum position reached by a random walker [5]. The presence of the reflector affects the asymptotic scaling properties of the walk and makes the critical exponent of the survival probability of the random walk vary continuously. Up to now, while many models exhibiting nonuniversal dynamical behaviors have been extensively investigated, there still remains uncertainty.

A few years ago, Krapivsky and Redner [7] investigated the survival probability of a random walker in a one-dimensional semi-infinite domain, \( x > x_0(t) = -\sqrt{At} \), where the borderline \( x = x_0 \) is the position of an absorbing boundary. In this model, \( x_0 \) was shown to be directly associated with the mean position of the walker as \( \langle x \rangle = \sqrt{At} \) by a coordinate transformation from \( (x, t) \) to \( (x' = x - x_0(t), t) \). This coordinate transformation makes the position of the absorbing boundary fixed at the origin and gives rise to a nonzero drift velocity \( v_d = \langle x \rangle /2t = (\sqrt{At}/2)t^{-1/2} \). In this case, richer and more interesting behavior arises, because the competition between \( \langle x \rangle = \sqrt{At} \) and the diffusion length \( \sqrt{Dt} \) plays an important role. Krapivsky and Redner calculated the survival probability of the walker by using a scaling ansatz \( P(x,t) \sim t^{-d+1/2} P(x/x_0) \) for the probability distribution of the walker. They showed that the critical exponent \( d \) of the survival probability varies continuously as \( A \) varies. In that case, the mean position

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of the walker $\langle x \rangle (= \sqrt{At})$ (or the nonzero drift velocity $v_d = (\sqrt{A}/2)t^{-1/2}$) with a power law in time gives the marginal perturbation in the random walk, so that the survival probability scales anomalously with a continuously varying exponent $\delta$ depending on its amplitude $A$.

In this paper, we study the anomalous behavior of the random walk with absorbing boundary at the origin and a movable wall on the other end by using the method analogous to that used by Krapivsky and Redner [7]. We first show that the probability distribution of the walker satisfies the same scaling ansatz as in Ref. 7. From the scaling ansatz, we show how the anomalous behavior of the continuously varying exponent $\delta$ arises. We also numerically find that the average drift velocity $v_d$ and the mean position $\langle x \rangle$ vary as $v_d = (C/2)t^{-1/2}$ and $\langle x \rangle = Ct^{1/2}$, respectively, where $C$ depends on the bias probability of the hopping on the moving wall. These power-law behaviors with $C$ depending on the bias probability are shown to be the physical origin of the marginal perturbation in this model as in Ref. 7. We also suggest an exact relation between $\langle x \rangle$ with an absorbing boundary and $\langle x \rangle$ without an absorbing boundary, which is also numerically confirmed. From this relation, we also accurately find the dependence of $C$ on the bias probability.

The random walk is defined on a one-dimensional lattice with an absorbing boundary at $x = 0$ and a movable wall at the maximum position that can be reached by a random walker. Initially the walker is at $x_0 = 1$, and the movable wall, whose position we denoted by $y$, is at $y_0 = 1$. If the walker is not at the same site as the moving wall, i.e., $x < y$, the walker jumps to either of two nearest neighbors with the probability $1/2$. However, when a walker is on the wall, i.e., $x = y$, either with probability $q$, both the walker and the wall move together one step to the right to a site not yet visited by the walker, or with probability $1 - q$, the walker moves to the left while the wall does not move. For $q = 1/2$, this model reduces to the normal random walk with an absorbing boundary.

We first measure the survival probability $S(t)$ for the walker to survive until time $t$, which scales algebraically in the long-time limit as

$$S(t) \sim t^{-\delta(q)}.$$  \hfill (1)

In Fig. 1, we display numerical data with various values of $q$ for the survival probability $S(t)$. As expected, the critical exponent $\delta(q)$ of the survival probability varies continuously with $q$, which reproduces the theoretical result of Dickman and ben-Avraham [5]: $\delta(q) = 1/2q - 1/2$ [the dotted curve in Fig. 1].

In order to understand the nonuniversal property appearing in this model, we study a probability distribution $P(x, t)$ that the walker is on $x$ at time $t$. To begin with, let us consider a normal random walk ($q = 1/2$ case) with an absorbing boundary at the origin. The probability distribution of the walker is exactly given by $[10,$

$$P(x, t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/24Dt} - e^{-(x+1)^2/4Dt},$$  \hfill (2)

where $D$ is the diffusion coefficient. For large $t$, Eq. (2) becomes

$$P(x, t) \sim \frac{1}{\sqrt{4\pi D t^{3/2}}} x e^{-x^2/4Dt} \sim t^{-\delta(1/2) - 1/2} \xi e^{-(\pi/4r)^2} x,$$  \hfill (3)

with a dimensionless variable $\xi \equiv x/\langle x \rangle$, where $\langle x \rangle$ is the mean position of the walker given by $\langle x \rangle = \int x P(x, t) dx / \int P(x, t) dx$. For $q = 1/2$, $\langle x \rangle = \sqrt{D\pi t^{1/2}}$ with $D = 1/2$ and $\delta(1/2) = 1/2$.

Since the probability distribution given by Eq. (3) satisfies Eq. (1) as $S(t) = \int_0^\infty P(x, t) dx = t^{-\delta(1/2)}$, Eq. (3) is a valid scaling relation for $q = 1/2$. It is, thus, a reasonable guess that the scaling relation for all other values of $q$ can generally be written as

$$P(x, t) = t^{-\delta(q) - 1/2} f(x/\langle x \rangle),$$  \hfill (4)

where $f(\xi)$ is a scaling function and $\delta(q) = 1/2q - 1/2$. As long as $\langle x \rangle$ of the walker scales as $t^{1/2}$, the scaling ansatz, Eq. (4), guarantees that the survival probability decays as $t^{-\delta(q)}$. In Fig. 2, the data for $P(x, t)$ for various time $t$ are shown to collapse well onto a single curve based on Eq. (4). The value of $\delta$ used in Fig. 2 is $\delta = 1/2q - 1/2$ with $q = 0.7$. From this result, it can be seen that the scaling relation, Eq. (4), holds for any values of $q$. Therefore, as long as $P(x, t)$ satisfies Eq. (4) and $\langle x \rangle = Ct^{1/2}$ with $C$ depending on $q$, it can be concluded that the exponent $\delta$ varies continuously with $q$. 

![Fig. 1. Plot of the critical exponent $\delta$ of the survival probability against $q$. The dotted curve is from the exact result given in Ref. 5.](image-url)
ability distribution \( C = 2^{5000} \), against \( x/t \).

Scaling and Survival Properties of Random Walks...

For this, we first show how \( C \) varies continuously with \( q \) to give a marginal perturbation in this model so that \( \delta \) is similar to the case in Ref. 7. Therefore, the bias is accurately estimated. For this, we first show how \( \langle x \rangle \) with the absorbing boundary is related to the mean position \( \langle x \rangle_0 \) without the absorbing boundary. First of all, one can reasonably assume that \( \langle x \rangle - \langle x \rangle_{q=1/2} = (x)_0 - (x)_{q=1/2} \), where \( (x)_{q=1/2} \) and \( (x)_{q=1} \) are the mean positions for the normal random walk with and without the absorbing boundary, respectively. Since \( (x)_{0,q=1/2} = 0 \), \( \langle x \rangle = (x)_{q=1/2} + \langle x \rangle_0 \). We numerically confirm this relation. Therefore, one can estimate \( \langle x \rangle \) exactly if one can calculate \( \langle x \rangle_0 \) exactly.

For the case without the absorbing boundary, we measure \( P(x,t) \) as shown in Fig. 4. Our numerical results indicate that the probability distribution \( P(x,t) \) has the form

\[
P(x,t) = \frac{A_L}{\sqrt{4\pi D_L t}} e^{-x^2/4D_Lt} \Theta(-x) + \frac{A_R}{\sqrt{4\pi D_R t}} e^{-x^2/4D_Rt} \Theta(x),
\]

where \( \Theta(x) \) is a step function. This prediction means that \( P(x,t) \) is a function patched from two Gaussian distributions with different widths: \( P(x,t) = P_L(x,t) \Theta(-x) + P_R(x,t) \Theta(x) \). On the positive \( x \) side, the width is greater than the width on the negative \( x \) side for \( q > 1/2 \). We also numerically find that the probabilities to find a walker on each side are given by \( \int_{-\infty}^{0} P_L(x,t) \, dx = 1 - q \) and \( \int_{0}^{\infty} P_R(x,t) \, dx = q \), respectively. From these relations, we predict that the amplitudes \( (A_L \) and \( A_R) \) and the diffusion coefficients \( (D_L \)

Numerically, we measure \( \langle x \rangle \) to see its \( q \) dependency. Figure 3 shows that \( \langle x \rangle \) grows as \( \langle x \rangle = Ct^{1/2} \), a power law in time, with \( C \) depending on \( q \). This scaling behavior is similar to the case in Ref. 7. Therefore, the bias hopping probability of the walker on the wall is expected to give a marginal perturbation in this model so that \( \delta \) varies continuously with \( q \).

Now, we want to explain how \( \langle x \rangle \) and \( v_d \) are accurately estimated. For this, we first show how \( \langle x \rangle \) with the ab-

Fig. 3. Plot of the mean position \( \langle x \rangle \) of the walker against time \( t \) for \( q = 0.5, 0.6, 0.7, \) and 0.8 (bottom to top). The lines represent the relation \( \langle x \rangle = Ct^{1/2} \), where Eq. (8) is used to estimate \( C \).

Fig. 4. The main plot shows data collapse of \( K_yP_y(x,t) t^{1/2} \) against \( x^2/4D_y t \) for \( q = 0.6, 0.7, \) and 0.8 on a semi-logarithmic scale where \( K_y = \sqrt{4\pi D_y} / A_y \) with \( y = L, R \). The inset (right top) shows a plot of \( P_L(x,t) t^{1/2} \) against \( x^2/t \) for \( q = 0.6, 0.7, \) and 0.8 (from left to right) at \( t = 100000 \). The inset (left bottom) is a similar plot of \( P_L(x,t) t^{1/2} \) against \( x^2/t \). Each plot shows that the data on a semi-logarithmic scale satisfy the Gaussian distribution of the random walker.
and $D_R$) in Eq. (5) can be written as

$$A_L = 2(1-q), \quad A_R = 2q,$$

$$D_L = \frac{1}{2}, \quad D_R = \frac{1}{2} \left( \frac{q}{1-q} \right)^2. \quad (6)$$

The insets in Fig. 4 show the numerical data for the probability distribution of the walker, $P_R(x,t)t^{1/2}$ (right top) and $P_L(x,t)t^{1/2}$ (left bottom), against $x^2/t$ at $t = 100\,000$ on each side. The slope in Fig. 4 for $P_R(x,t)$ is $1/4D_R$, which is a function of $q$ [see Eq. (6)], but for $P_L(x,t)$, it is $1/4D_L = 1/2$, which is independent of $q$. The main plot in Fig. 4 shows that when we use the values in Eq. (6), our data for various $q$ collapse very well onto a single curve based on Eq. (5).

Equation (5) with Eq. (6) allows us to calculate the mean position of the walker for the case without the absorbing boundary:

$$\langle x \rangle_0 = \sqrt{\frac{2}{\pi}} \frac{2q-1}{1-q} t^{1/2} \equiv C_0 t^{1/2}. \quad (7)$$

The inset in Fig. 5 shows that $\langle x \rangle_0$ grows as a power law in time as $\langle x \rangle_0 \propto C_0 t^{1/2}$. As shown in Fig. 5, our numerical results for $C_0$ as a function of $q$ are well consistent with the prediction, $C_0 = \sqrt{2/\pi} (2q-1)/(1-q)$ from Eq. (7). Finally, using the relation $\langle x \rangle = \langle x \rangle_{q=1/2}^0 + \langle x \rangle_0$, one can obtain $\langle x \rangle$ for this model with the absorbing boundary as

$$\langle x \rangle = \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi} \frac{2q-1}{1-q}} \right) t^{1/2} \equiv Ct^{1/2}. \quad (8)$$

From Eq. (8), the average drift velocity $v_d$ can also be calculated as $v_d = (C/2)t^{-1/2}$.

In summary, we study a one-dimensional lattice random walk with an absorbing boundary at the origin and a movable wall on the other end. We confirm that the probability distribution $P(x,t)$ of the walker follows the scaling ansatz $P(x,t) = t^{-\delta(q)-1/2} f(x/\langle x \rangle)$ and that the survival probability scales anomalously with a continuously varying critical exponent $\delta(q)$, as long as $\langle x \rangle$ scales as $t^{1/2}$. We also find the exact relation between $\langle x \rangle$ and $v_d$ with the absorbing boundary and those without the absorbing boundary. From the relation, we can accurately estimate the values, $\langle x \rangle$ and $v_d$, for the case with the absorbing boundary. Although the random walks we discuss in this paper cannot exactly describe the boundaries of the active region in systems exhibiting an absorbing phase transition with infinitely many absorbing states, we find that the bias hopping probability on the wall changes the scaling behavior of the survival probability anomalously.

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