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Classification of transport backbones of complex networks

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Transport properties in random and scale-free (SF) networks are studied by analyzing the betweenness centrality (BC) distribution $P(B)$ in the minimum spanning trees (MSTs) and infinite incipient percolation clusters (IIPCs) of the networks. It is found that $P(B)$ in MSTs scales as $P(B) \sim B^{-\delta}$. The obtained values of δ are classified into two different categories, $\delta \simeq 1.6$ and $\delta \simeq 2.0$. Using the mapping between BC and the branch size of tree structures, it is proved that δ in MSTs which are close to critical trees is 1.6. In contrast, δ in MSTs which are supercritical trees is shown to be 2.0. We also find $\delta = 1.5$ in IIPCs, which is a natural result because IIPC is physically critical. Based on the results in MSTs, a physical reason why $\delta \geqslant 2$ in the original networks is suggested.

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Ubiquitousness of weblike structures has triggered a huge number of studies on complex networks [1–5]. Early studies on structural properties of complex networks have successfully uncovered many interesting topological features. Among them, the most important finding was that those weblike structures share some common properties. Small-worldness is one of the well-known common features of various networks. For more quantitative analyses of such complex structures, degree distribution, clustering coefficient, degree-degree correlation, betweenness centrality, etc., were studied in detail. In addition to such topological properties, the dynamical properties on complex networks such as transport [6,7], synchronization [8], and epidemic spreading [9] were also investigated. The interplay between the dynamical and topological properties of complex networks is also crucial to understand various dynamical phenomena in complex networks [10]. From this point of view, understanding the transport property in complex networks is very important to uncover the underlying evolutionary mechanisms in diverse disciplines of sciences, including physics, biology, and sociology. Examples include metabolic fluxes in metabolic cycle [11], information flow between individuals [12], and diffusive dynamics on complex networks [7]. Like many other dynamical properties, transport in complex networks is also known to be crucially affected by the underlying structures such as the self-similarity of network [7].

Recently, for a more realistic description of real networks, much effort has been focused on weighted networks [13,14] in which a weight is assigned to each link. An example of weight is the transportation cost between cities in transportation networks. In such weighted networks, reducing the transportation cost is one of the important factors to determine the route to move from one node to the others. Thus, the minimum spanning tree (MST), which connects all nodes with the minimum total weights, is regarded as an important transport backbone of the network. Another important transport backbone of the network is the infinite incipient percolation cluster (IIPC).

The transport in various networks was quantitatively studied [6,15–18] based on the scaling behavior of the betweenness centrality (BC) distribution $P(B)$. BC of a node is the number of shortest paths in a network passing through the node [15]. Thus, $P(B)$ is an important quantity which characterizes the transport in complex networks. Recently, $P(B)$'s in MSTs and IIPCs of both random and scale-free (SF) networks were studied to characterize the transport property of the complex networks, where MSTs were constructed by a random weight assignment scheme [6]. Based on the numerical measurements, $P(B)$ was argued to satisfy a power law,

$$P(B) \sim B^{-\delta}, \tag{1}$$

with $\delta = 1.6$–1.7 in MSTs and $\delta \simeq 1.2$ in IIPCs [6]. These values of δ significantly deviate from the values $\delta \simeq 2.0$ and $\delta = \infty$ measured in the original SF and random networks, respectively [6,16–18]. This implies that the transport property in MST or in IIPC differs from that in original networks. Therefore, to understand the physical property and to provide a better strategy for the enhancement of the global transport, it is crucial to find the physical origin of such differences in the various networks.

In this Rapid Communication we show that the transport backbones are divided into two fundamental classes based on the measured value of δ. From the relation between BC and the branch size of the transport backbone, it is found that $\delta = 3/2$ when the transport backbone is a critical tree and $\delta = 2$ when the backbone is a supercritical tree [19,20]. This result clearly indicates that $\delta > 1.2$ in IIPCs, which contradicts the numerical observation of Wu *et al.* [6]. Similar results were found in the hierarchical distribution of subcommunities within communities [21]. In addition, it is found that the obtained value of δ in MSTs which are physically very close to a critical tree is slightly larger than $3/2$. Such a deviation is explained by the real tree structure of the MSTs. Additionally, based on the insights obtained from the results in transport backbones, we also provide a possible origin of $\delta \geqslant 2.0$ in various original networks [16–18].

To study the transport in MSTs and IIPCs, random and SF networks are first constructed. For a construction of SF networks with a prescribed degree distribution, $P(k) \sim k^{-\gamma}$, the static model is used [16]. In the static model, the weight

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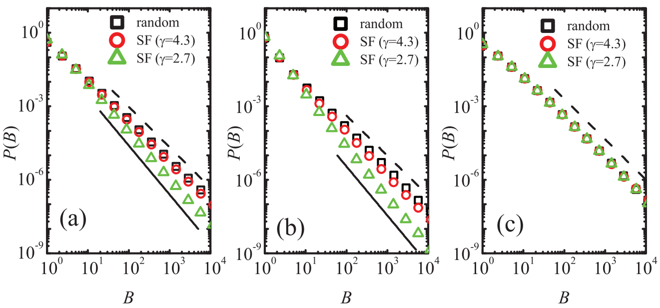

FIG. 1. (Color online) Plots of $P(B)$ against B in MSTs extracted from the random and SF networks. (a) MSTs are from the networks where the weight w_{ij} is random. (b) MSTs are from networks with $w_{ij} = 1/(k_i k_j)$, and (c) MSTs are from those with $w_{ij} = k_i k_j$. Solid lines correspond to the relation $P(B) \sim B^{-2.0}$, and dashed lines denote the relation $P(B) \sim B^{-1.6}$.

$W_i = i^{-\alpha}$ is assigned to each node i $(i = 1,\dots,N)$. Then, two different nodes (i,j) are chosen by using the probabilities $W_i/\sum_k W_k$ and $W_j/\sum_k W_k$ and are connected if the two nodes (i,j) are not already linked. In this network, γ and α satisfy the relation $\gamma = (1+\alpha)/\alpha$. To construct random networks, α is set to be zero in the static model. Next, the transport backbones of the constructed networks are extracted. IIPC is obtained in the following way. Starting from the given network, a link is randomly chosen to be removed. Then, $\kappa \equiv \langle k^2\rangle/\langle k\rangle$ is checked. If $\kappa > 2$, the removing process of links is continued. Otherwise, the removing process is stopped because the largest cluster in the network becomes IIPC [6] at $\kappa = 2$. To extract MSTs of constructed SF and random networks, a weight w_{ij} to each link between nodes i and j is assigned by three assignment schemes. The simplest one is the random weight assignment scheme in which w_{ij} is a random number in the interval [0,1]. In the second weight assignment scheme, $w_{ij} = 1/(k_i k_j)$. An example of this kind of weighted networks is a scientist collaboration network [14]. In the third weight assignment scheme, $w_{ij} = k_i k_j$. An example of the third kind is the airport network [14]. Then, MSTs are extracted by using Prim's algorithm [22].

To find the characteristics of the transport backbones, the probability distributions of BC $P(B)$ in MSTs and IIPCs are measured. BC of node v, B_v, is defined as [16,23,24]

$$B_v \equiv \sum_{i=1}^{N}\sum_{j=i}^{N}\frac{S_{i,v,j}}{S_{i,j}}. \tag{2}$$

Here $S_{i,j}$ is the total number of the shortest paths from node i to j, and $S_{i,v,j}$ is the number of the shortest paths from node i to j which pass through node v [24]. Due to the complexity in the computation of B several algorithms have been suggested [16,23]. In our analysis, we use Newman's algorithm to measure $P(B)$ [23]. $P(B)$'s measured in MSTs extracted from the networks with $N = 10^5$ are displayed in Fig. 1. $P(B)$'s satisfy the power law in Eq. (1) when $B > 100$. Furthermore, as shown in Figs. 1(a) and 1(b), the measured δ is either 1.6(1) or 2.0(1) when w_{ij} is random or $w_{ij} = 1/(k_i k_j)$. $\delta = 1.6(1)$ in MSTs both from random networks and from SF networks with $\gamma > 3$, whereas $\delta = 2.0(1)$ in MSTs from SF networks with $\gamma < 3$. On the other hand, the measured δ in MSTs from the networks with $w_{ij} = k_i k_j$ is 1.6(1), regardless of γ, as shown in Fig. 1(c). Therefore, MSTs can be classified into two different categories. The first one is MSTs with $\delta \simeq 1.6$, and the other one is MSTs with $\delta \simeq 2.0$.

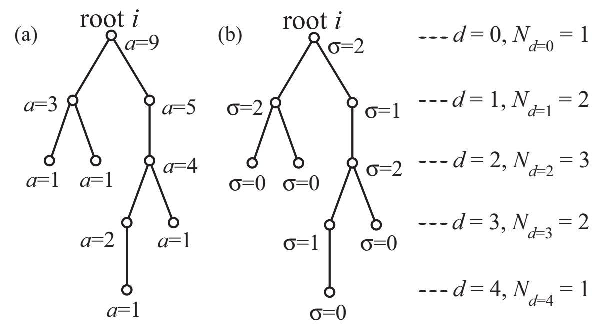

FIG. 2. Schematic illustrations of (a) the branch size a of each node in a hierarchical tree rooted at node i and (b) the number of new branches σ at each node. d represents the distance to a node from the root i, and N_d is the number of nodes at distance d.

The physical origin of why only two different categories exist can be explained by the mapping between BC and the branch size of tree structures [19,20]. The branch size is an important quantity to characterize a hierarchical tree [see Fig. 2(a)]. The branch size of node v, a_{vi}, in a hierarchical tree rooted at node i is defined as the number of nodes in the subtree rooted at node v [see Fig. 2(a)] [25]. It is well known that the branch size distribution follows a power law [19,20],

$$P(a) \sim a^{-\tau}. \tag{3}$$

The exponent τ is known to be $\tau = 3/2$ in a critical tree and $\tau = 2$ in a supercritical tree [19,20]. Since the tree does not have a loop, the branch size a_{vi} becomes exactly the same as the number of the shortest paths from the root i to every node passing through node v, i.e., $a_{vi} = \sum_{j=1}^{N} S_{i,v,j}$. Therefore, from Eq. (2) BC of v in a tree is written as

$$B_v = \frac{1}{2}\left(1 + \sum_{i=1}^{N} a_{vi}\right) \tag{4}$$

because $S_{i,j} = 1$ in any tree. From Eqs. (3) and (4) and Ramsay's rule for the distribution of sums of independent identically distributed Pareto variables [26], we obtain

$$\delta = \tau. \tag{5}$$

Therefore, if the transport backbone is a critical tree, then $\delta = 3/2$, and $\delta = 2$ if the transport backbone is supercritical.

A tree becomes critical when the branching ratio is $\langle\sigma\rangle = 1$. On the other hand, a tree becomes supercritical when $\langle\sigma\rangle > 1$ [see Fig. 2(b)] [19]. However, direct measurement of $\langle\sigma\rangle$ from an already-constructed finite tree is not trivial because of a strong finite-size effect. In order to investigate the branching ratio of MSTs, an effective branching ratio σ_d defined as $\sigma_d \equiv N_{d+1}/N_d$ is first measured in exact critical and supercritical trees. Here d represents the distance to a node from the root i, and N_d is the number of nodes at distance d [see Fig. 2(b)].

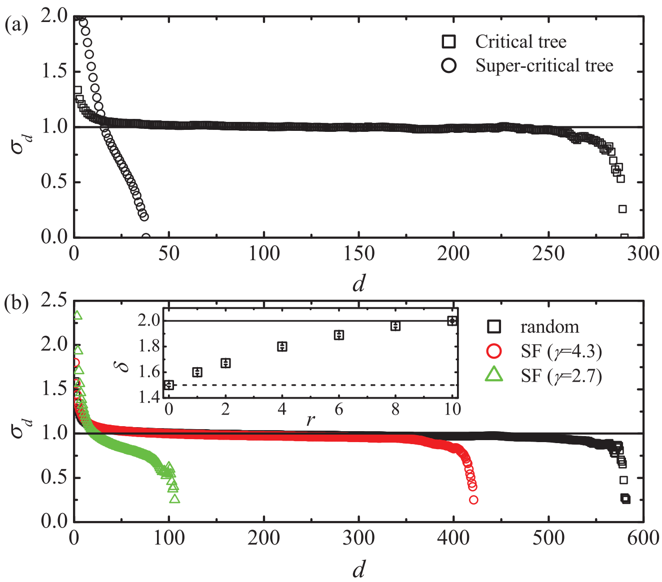

FIG. 3. (Color online) (a) Plots of σ_d against d for an exact critical tree (squares) and an exact supercritical tree (circles) with $N = 10^4$. σ_d for the critical tree shows a long plateau around $\sigma_d = 1$ (solid line). (b) Plots of σ_d measured in MSTs extracted from the networks with $N = 10^5$ using random w_{ij}. σ_d's in MSTs from random networks and SF networks with $\gamma > 3$ show a long plateau around $\sigma_d = 1$. In contrast, σ_d in MST from SF networks with $\gamma < 3$ does not have any plateau. The inset shows δ against the ratio of the size of supercritical trees to that of critical trees r.

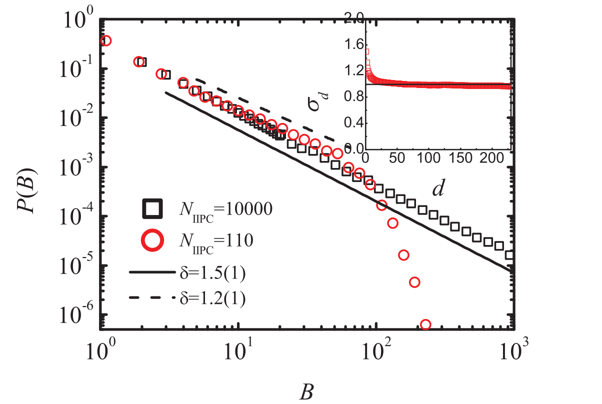

FIG. 4. (Color online) Plots of $P(B)$ measured in the IIPC from the random networks. The dashed line denotes the relation $P(B) \sim B^{-1.25}$, and the solid line denotes $P(B) \sim B^{-1.5}$. The inset is a plot of σ_d. The solid line denotes $\sigma_d = 1$ as a guide to the eye.

In Fig. 3(a) σ_d's for exact critical and supercritical trees are displayed. The critical tree has a much larger diameter than the supercritical tree for the same N, and σ_d of the critical tree has a long plateau around $\sigma_d = 1$. On the other hand, the supercritical tree has a relatively small diameter, and σ_d of the supercritical tree decays much faster than that of the critical tree without any plateau around $\sigma_d = 1$. In Fig. 3(b) we show σ_d in MSTs extracted from the networks with random w_{ij} as an example. σ_d's in MSTs from random networks and SF networks with $\gamma > 3$ show that the MSTs have a relatively long diameter and a long plateau around $\sigma_d = 1$ as the exact critical tree. On the other hand, σ_d in MSTs from SF networks with $\gamma < 3$ does not have such a plateau. These results indicate that MSTs from random networks and SF networks with $\gamma > 3$ are close to the critical tree, but that from a SF network with $\gamma < 3$ becomes supercritical. From the results in Fig. 1(a), it is identified that $P(B) \sim B^{-1.6}$ for $\sigma_d = 1$. In contrast, $P(B) \sim B^{-2.0}$ when σ_d shows the supercritical behavior. It is confirmed that $P(B)$ shows the same behavior in MSTs from the networks with $w_{ij} = 1/(k_i k_j)$ and $w_{ij} = k_i k_j$.

When σ_d of MST behaves like a critical tree, the obtained value of δ is close to $\delta = 3/2$ but slightly deviates from $\delta = 3/2$. To find the origin of such a slight deviation, $P(B)$ is measured on a combined tree in which a critical tree is connected to supercritical trees. In the combined tree, some of dangling nodes of the critical tree are randomly chosen. Then, each of the randomly chosen nodes is connected to a supercritical tree. In the inset of Fig. 3(b), the measured δ's against the ratio of the size of the supercritical trees to that of the critical tree r are displayed. From the data it is found that δ for the combined tree increases as the fraction of nodes in the part with supercritical trees increases and approaches $\delta = 2.0$. So the slight deviation can be explained by the fact that the MST with $\delta = 1.6$ is very close to an exact critical tree but has a small fraction for the supercritical tree.

Since IIPC is obtained at the percolation threshold, IIPC should physically be a critical tree. As shown in the inset of Fig. 4, σ_d in IIPC from the random network shows the characteristics of a critical tree ($\langle\sigma\rangle \simeq 1$). Additionally, $P(a)$ on the IIPC is found to satisfy $P(a) \sim a^{-\tau}$ with $\tau = 1.5$. These results clearly prove that the IIPC is a critical tree. Thus, it is not theoretically possible that $\delta \simeq 1.25$ for the IIPC as argued in Ref. [6]. As shown in Fig. 4, we find that $\delta \simeq 1.25$ in the small IIPC with the number of nodes $N_{IIPC} = 110$, which is obtained from the random network with $N = 2^{13}$. However, as N_{IIPC} gets larger, δ approaches 1.5. In the IIPC with $N_{IIPC} = 10^4$ which is obtained from the random network with $N = 10^6$, $\delta = 1.5(1)$ is exactly measured (see Fig. 4). This clearly shows that $\delta \simeq 1.25$ for IIPC was underestimated due to the finite-size effect [6].

Finally, we want to discuss δ in original networks themselves in the light of the insights obtained from the results in transport backbones. In many cases it is well known that the local tree approximation in sparsely connected networks predicts the correct topological and dynamical behaviors of networks, such as percolation [5]. Thus, if the local tree approximation is valid, then the possible value of δ in original networks should be in the range $3/2 \leqslant \delta \leqslant 2$. However, the value of δ obtained from the original network was generally known to be $\delta \geqslant 2$ [16–18]. If the effect of loops in networks is strong enough, then the tree approximation would not be valid any more. Thus, to investigate the effect of loops, additional links are added to the critical tree. As the number of additional links is increased, the structure of the modified critical tree with loops approaches that of a random network. In Fig. 5 we compare $P(B)$'s measured on the modified critical trees. Theoretically, $\delta = 3/2$ in the exact critical tree [19]. However, when a small number of links are added to the critical tree, $P(B)$ is drastically changed and exponentially decays, which corresponds to $\delta = \infty$ (see in Fig. 5). This result may explain

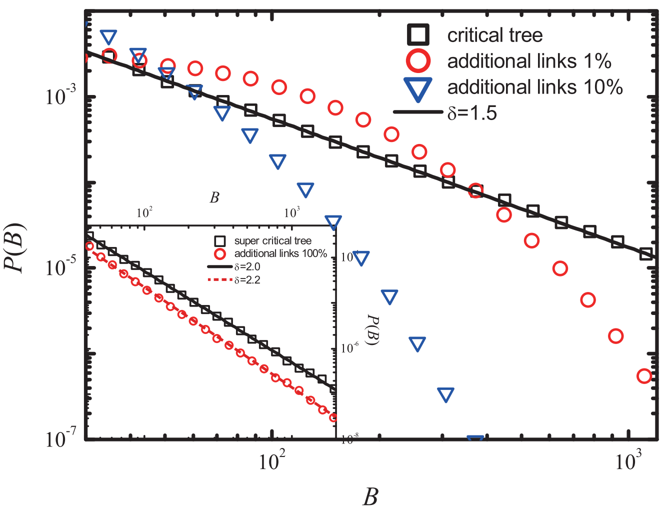

FIG. 5. (Color online) Plots of $P(B)$ measured from the modified critical tree with additional links. The density of additional links is changed from 0% to 10%. Here the density of additional links is defined as the ratio of the number of additional links to the number of nodes ($N = 10^5$). The solid line represents $\delta = 1.5$. The inset shows $P(B)$ measured from the modified supercritical tree with additional links. The density of additional links is changed from 0% to 100%. The solid line represents $\delta = 2.0$, and the dashed line denotes $\delta = 2.2$.

physically why $\delta = \infty$ in the random networks [16,17]. On the other hand, the effect of additional links in the exact supercritical tree is relatively weak. When a considerably large number of additional links are added to the supercritical tree, $P(B)$ still satisfies the power-law equation (1), and δ is close to $\delta = 2.0$ but slightly deviates from $\delta = 2.0$ (see the inset of Fig. 5). This also may explain why $\delta \simeq 2$ in SF networks with $\gamma < 3$ [16,17].

In summary, we study the transport property in complex networks through the scaling behavior of $P(B)$ on two different transport backbones, MST and IIPC. From the numerical analyses, it is found that $P(B)$ measured on the transport backbones scales as $P(B) \sim B^{-\delta}$. The obtained values of δ on the transport backbones are classified into two different categories, $\delta \simeq 1.6$ and $\delta \simeq 2.0$. From the relation between BC and the branch size in tree structures, it is shown that $\delta \simeq 1.6$ in critical MSTs and $\delta \simeq 2.0$ in supercritical MSTs. This result clearly shows that the known value $\delta \simeq 1.25$ in IIPC [6] was underestimated due to the strong finite-size effect. We also find that the degree-degree correlation in the original networks does not affect the scaling behavior of $P(B)$ on the transport backbones. Finally, we provide a possible origin of $\delta \geqslant 2.0$ in various original networks.

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