Bond-site duality and nature of the explosive-percolation phase transition on a two-dimensional lattice

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To establish the bond-site duality of explosive percolations in two dimensions, the site and bond explosive-percolation models are carefully defined on a square lattice. By studying the cluster distribution function and the behavior of the second largest cluster, it is shown that the duality in which the transition is discontinuous exists for the pairs of the site model and the corresponding bond model which relatively enhances the intrabond occupation. In contrast the intrabond-suppressed models which have no corresponding site models undergo a continuous transition and satisfy the normal scaling ansatz as ordinary percolation.

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I. INTRODUCTION

Recently, the Achlioptas process (AP) [1], which was suggested to show a supposedly first order transition on the complete graph, triggered intensive studies on explosive percolations [2–7]. However subsequent studies have proved that the transition in the AP on the complete graph is continuous [4–7]. We have also shown that the AP on the Bethe lattice shows a continuous transition [8]. Therefore the transition in the original AP is physically established to be continuous in the mean-field level or in high dimensions.

Until now studies on the AP have been done mainly on the complete graph. Even though there are some studies on the AP in two dimensions (2D) [3,7,9–12], the transition nature in lower dimensions is still not fully understood. For example, the bond percolation under the AP with a product rule was first argued to show a discontinuous transition [3,9]. However, based on the measurement of the largest cluster distribution, Grassberger *et al.* [7] argued that the bond percolation with the same product rule in 2D undergoes continuous transition [7]. The site percolation under the AP with a product rule in 2D has been proved to undergo a discontinuous transition based on a detailed analysis of cluster size distribution and hysteresis [11]. In contrast, Bastas *et al.* argued that the site percolation under the AP with a sum rule in 2D undergoes continuous transition based on the finite-size scaling analysis with relatively small system sizes [12]. Those controversies also show that we cannot simply exclude the possibility that the transition nature of the AP in the mean-field limit is different from that in lower-dimensional systems, such as the well-studied Potts model [13].

Such controversies [3,7,9,11,12] also imply that there does not seem to exist a bond-site duality among explosive-percolation models on 2D lattices, unlike ordinary percolation [14,15]. Here bond-site duality means that a bond percolation model has the same transition nature or belongs to the same universality class as the corresponding site percolation model except for the properties depending on the details of the models, such as transition probability p_c. Moreover, if the transition is truly discontinuous, then determination of critical exponents from finite-size scaling analysis as in Ref. [12] has no physical meaning, as we already addressed in Ref. [11]. Thus, resolving such controversies by establishing the bond-site duality in 2D is theoretically very important and interesting.

The controversies should come from ambiguities in the definitions of explosive-percolation models on lattices. Therefore it is very important to make clear definitions of models with various growth rules for the AP. There can be six kinds of models on a 2D square lattice. Among them, two pairs are bond models. One pair consists of the bond models which physically enhance the occupation of intracluster bonds. The other pair consists of bond models which relatively suppress the occupation of intrabonds.

In site percolation, there cannot be a distinction between interbonds and intrabonds in a cluster because the occupation of any empty site always increases the size of the cluster. So there is no ambiguity in the definition of site models as in bond models. As we shall see, the intrabond-enhanced models and the corresponding site models show the bond-site duality in which the transition is discontinuous. In contrast the intrabond-suppressed models show a continuous transition. Physically, there should be no site model corresponding to such intrabond-suppressed models.

II. MODELS

There are two fundamental percolation models on lattices [14]. One is the site percolation model, and the other is the bond percolation model. In the site percolation model, there is no new site occupation which does not change the size of clusters. Under the AP, two vacant sites A and B are randomly selected. Let $\{s_{A_i}\}$ ($\{s_{B_j}\}$) be the sizes of n_A (n_B) clusters which would be connected by occupying site A (B). In the square lattice n_A (n_B) is at most 4. In the site model with a product rule (SP model) site A is occupied if $1 \times \prod_{i=1}^{n_A} s_{A_i} < 1 \times \prod_{i=1}^{n_B} s_{B_i}$. Otherwise, site B is occupied. Similarly, in the site model with a sum rule (SS model) site A is occupied if $(1 + \sum_{i=1}^{n_A} s_{A_i}) < (1 + \sum_{j=1}^{n_B} s_{B_j})$.

We define four bond percolation models under AP. The first two unoccupied bonds a and b are selected randomly. If bond a is an interbond, then it connects two different clusters of sizes s_{a1} and s_{a2}. Then under a product rule product ξ_a for bond a is clearly defined as $\xi_a \equiv s_{a1} \times s_{a2}$ without any ambiguity. If

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the bond is an intrabond, it internally connects two sites in the same cluster. Then ξ_a can be defined in two different ways. One is $\xi_a \equiv s_{a1} \times 1$ [bond product type 1 (BP1) model]. The other is $\xi_a \equiv s_{a1} \times s_{a1}$ [bond product type 2 (BP2) model]. The product ξ_b for bond b is similarly defined. Bond a will be occupied if $\xi_a < \xi_b$. Otherwise, bond b will be occupied. Therefore the two bond product models, BP1 and BP2, come from the ambiguity in defining the product for the selected intrabond. Similarly, we can define two kinds of bond models with a sum rule. The sum σ_a for interbond a is defined clearly as $\sigma_a \equiv s_{a1} + s_{a2}$. In contrast σ_a for intrabond a can also be defined in two-different ways: $\sigma_a \equiv s_{a1} + 0$ [bond sum type 1 (BS1) model] or $\sigma_a \equiv s_{a1} + s_{a1}$ [bond sum type 2 (BS2) model]. Then bond a will be occupied if $\sigma_a < \sigma_b$.

The physical meaning of the models is that type 1 models (BP1 and BS1) relatively enhance the intrabond occupation, whereas type 2 models (BP2 and BS2) suppress the intrabond occupation. Thus if bond-site duality exists, it should be between type 1 bond models and site models. As we shall see, the duality exists for the pair of BP1 and SP models and the pair of BS1 and SS models. BP2 and BS2 have no corresponding site models for the duality.

Until now, only the BP2 model has been studied for the explosive bond percolation model [3,7,9]. In Ref. [9] the BP2 model was argued to show a discontinuous transition, whereas in Ref. [7] the same model was argued to undergo a continuous transition. The BP1, BS1, and BS2 models have not been studied. Both SP and SS have been studied as explosive site percolation models. The SP model [11] has been proved to show a discontinuous transition, whereas the SS model [12] was argued to show a continuous transition based on numerical studies of relatively small systems.

In the following sections, based on more precise simulations with larger system sizes, we will show the existence of the duality between bond and site percolation AP models, as well as the characterization of the transition nature of each model. For this purpose, we use $L = 512$–8192, and the data are averaged over 10^6 independent runs.

III. CLUSTER SIZE DISTRIBUTION

To understand the transition nature of the percolation physically, the cluster size distribution should be understood first. The cluster size distribution $P_s(p)$ at an occupation probability p of a bond (or site) is the probability that an occupied site belongs to a cluster which has s sites. It has been shown that $P_s(p)$ provides an excellent method to determine p_c as well as the transition nature for an ordinary percolation (OP) [14] and explosive percolations [5,11]. Thus, we first investigate $P_s(p)$ for each model to obtain p_c and the transition nature physically. When $p < p_c$, P_s for OP decays exponentially as s increases. This means that the probability of finding a large cluster vanishes exponentially. On the other hand, when $p > p_c$, there exists a macroscopically large cluster (LC), and P_s for finite s also decays exponentially with a peak for the LC. At $p = p_c$ it is well known that P_s satisfies a power law, $P_s(p_c) \sim s^{-\delta}$, with $\delta \simeq 1.055$ for OP.

In contrast, P_s for the SP model in Fig. 1(a) is completely different from that for OP, as shown in Ref. [11]. When $p < p_c$, P_s for the SP model has a stable hump in the tail as $p \to p_c^-$

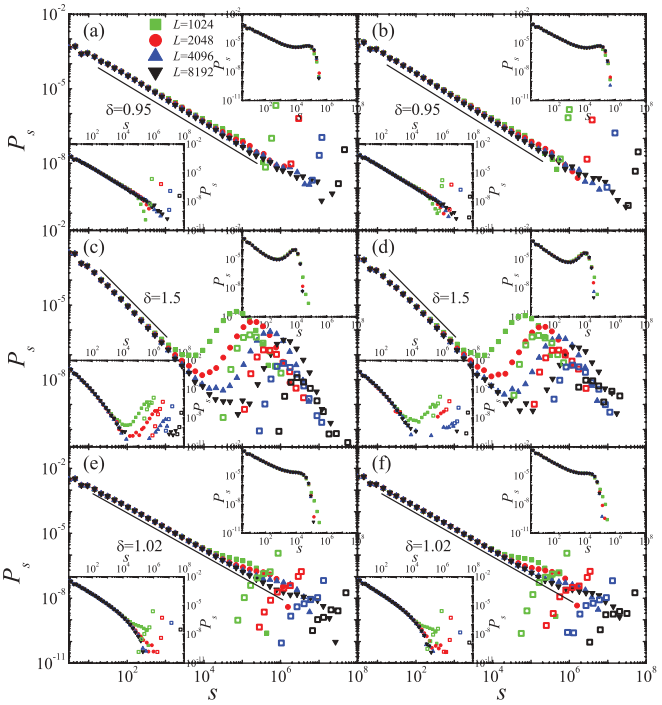

FIG. 1. (Color online) Plots of $P_s(p)$ against s for various models. Open symbols denote P_s for the largest cluster on the lattice with linear size L. (a) SP model at $p = 0.7723(7)$ ($\approx p_c$). Insets: $p = 0.75$ ($< p_c$; top inset) and $p = 0.78$ ($> p_c$; bottom inset). The line with $\delta = 0.95$ indicates the relation $P_s \sim s^{-0.95}$. (b) The BP1 model at $p = 0.6937(8)$ ($\approx p_c$). Insets: $p = 0.67$ ($< p_c$; top inset) and $p = 0.70$ ($> p_c$; bottom inset). The line is the same as in (a). (c) The BS1 model at $p = 0.5979(4)$ ($\approx p_c$). Insets: $p = 0.590$ ($< p_c$; upper inset) and $p = 0.605$ ($> p_c$; bottom inset). The line denotes the relation $P_s \sim s^{-1.5}$. (d) The SS model at $p = 0.6916(5)$ ($\approx p_c$). Insets: P_s at $p = 0.685$ ($< p_c$; top inset) and $p = 0.695$ ($> p_c$; bottom inset). The line is the same as in (c). (e) The BP2 model at $p = 0.5266(1)$ ($\approx p_c$). Insets: $p = 0.5250$ ($< p_c$; top inset) and $p = 0.5270$ ($> p_c$; bottom inset). The line denotes the relation $P_s \sim s^{-1.02}$. (f) The BS2 model at $p = 0.5270(1)$ ($\approx p_c$). Insets: $p = 0.5250$ ($< p_c$; top inset) and $p = 0.5275$ ($> p_c$; bottom inset). The line is the same as in (e).

[5,11]. The behavior of the hump for the SP model has been studied in detail in Ref. [11]. On the other hand, at $p \simeq p_c$, P_s for a LC starts to become detached from the continuous distribution of P_s for finite s, which satisfies $P_s \sim s^{-\delta}$ with $\delta = 0.95(1)$ (< 1) [16]. Such power-law behavior for finite s is observed for sufficiently large p ($> p_c$) [see the bottom inset in Fig. 1(a)] [11]. Based on this typical behavior of $P_s(p)$, we determined p_c for the SP model [11]. Since the hump contribution for $p < p_c$ does not depend on the lattice size L and $\delta < 1$ at p_c, there should be many stable large (but still microscopic) clusters before transition, which strongly indicates a discontinuous transition for the SP model [11]. To determine p_c and find the transition nature for other models, we now analyze P_s as in Ref. [11].

In Fig. 1(b) we display $P_s(p)$ for the BP1 model. The data for $p < p_c$ in the top inset in Fig. 1(b) clearly shows that the hump in $P_s(p < p_c)$ does not depend on L as for the SP model in Fig. 1(a). At $p \simeq p_c$, P_s for LC starts to become detached from the continuous distribution of P_s, as shown in the main plot of Fig. 1(b). From the best fit of the data for finite s, we

obtain $P_s \sim s^{\delta}$ with $\delta \simeq 0.95(1)$ (<1). For $p > p_c$ we find that P_s for finite s still satisfies the power law with $\delta < 1$ (see the bottom inset). These behaviors of $P_s(p)$ for the BP1 model exactly coincide with that for the SP model, which is known to undergo discontinuous transition [11]. The only difference between the BP1 model and the SP model is in the value of p_c: $p_c \simeq 0.6937(8)$ for the BP1 model and $p_c \simeq 0.7723(7)$ for the SP model. The coincidence of P_s between the BP1 model and the SP model physically means that there is a bond-site duality under the AP with the product rule if the bond model enhances the intrabond occupation.

Figures 1(c) and 1(d) show P_s for the BS1 and SS models. By assuming the relation $P_s \sim s^{\delta}$ only for small s at $p \simeq p_c$, we obtain $\delta \simeq 1.5$ for the BS1 and SS models. However, as depicted in Figs. 1(c) and 1(d), P_s for the BS1 and SS models seems to substantially deviate from the power law due to a main contribution from the hump distribution for large s. Even for the value of p_c at which P_s for LC starts to become detached from the seemingly power-law-like regime, the hump still exists. The contribution of the hump distribution part to $\sum_s^{L^2} P(s) = p$ for site percolation [or $\sum_{s=1}^{L^2} P(s) = 1$ for bond percolation] even at $p = p_c$ is more than 90% [17]. This means that the contribution of the power-law-like regime for small clusters to $\sum_s P(s)$ is trivial. This result implies the absence of the singular behavior, $P_s(p) \sim s^{-\delta}$ for $s \to \infty$ at p_c, which is the intrinsic property of the continuous transition. Moreover in both BS and SS models the hump distribution for $p < p_c$ does not depend on the lattice size L (see the top insets). This anomalous behavior suggests that there exists a type of powder keg [18] in the BS1 and SS models, unlike in the fully connected networks [6]. In addition, the existence of the hump distribution even for $p > p_c$ (see the bottom insets) implies the existence of multiple stable macroscopic clusters for $p \geqslant p_c$ in the BS1 and SS models. Furthermore, P_s for the BS1 model [Fig. 1(c)] is nearly identical to P_s for the SS model [Fig. 1(d)] as in the case for the BP1 and SP models. The coincidence of P_s between the BS1 and SS models except for the value of p_c also implies that there is bond-site duality under the AP with the sum rule if the bond model enhances the intrabond occupation.

In Figs. 1(e) and 1(f) we display P_s for the BP2 and BS2 models. P_s for the BP2 and BS2 models is physically different from P_s for the BP1 and BS1 models. Even though P_s for the BP2 and BS2 models has a hump in the tail when $p < p_c$ (top inset), this hump vanishes in the limit $L \to \infty$, and P_s at p_c shows the power-law singularity $P_s(p) \sim s^{-\delta}$ with $\delta = 1.02(1) > 1$. As we increase p further to $p > p_c$, P_s for finite s decays exponentially as in OP (see the bottom insets for $p > p_c$). This P_s behavior is physically the same as that in OP. As for OP, P_s or n_s ($\equiv P_s/s$) at p_c for the percolation with continuous transition satisfies the scaling relation [14]

$$P_s = s^{-\delta} f(s/L^{1/\sigma\nu})[n_s = s^{-\tau} f(s/L^{1/\sigma\nu})], \qquad (1)$$

where $\delta = \tau - 1$ and ν is the correlation length critical exponent. As shown in Figs. 2(a) and 2(b), P_s (n_s) for both the BP2 and BS2 models satisfies scaling relation (1) with $\delta = 1.02$ ($\tau = 2.02$) and $\nu\sigma = 0.51$ very well. P_s in Figs. 1(e), 1(f), 2(a), and 2(b) strongly suggests that both the BS2 and BP2 models show continuous transition. Our estimation of

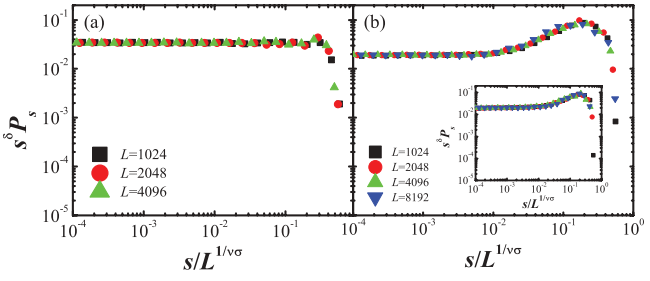

FIG. 2. (Color online) (a) Scaling collapse for relation (1) of the BP2 model and (b) the collapse of the BS2 model. The exponents in (a) and (b) are $\delta = 1.02$ and $\nu\sigma = 0.51$. Inset: scaling collapse for OP with $\delta = 1.05$ and $\nu\sigma = 0.53$.

$p_c = 0.5266(1)$ for the BP2 model is nearly the same as those in Refs. [7,9], even though the two references were contradictory to each other in transition nature.

IV. FINITE-SIZE SCALING ANALYSIS FOR THE BP2 AND BS2 MODELS

From τ and σ obtained from Eq. (1), one can calculate the critical exponents β, γ, and ν, which must be identical to the values evaluated from the finite-size scaling (FSS) properties of the average-size $S(p,L)$ of the finite clusters and the order parameter $P_\infty(p,L)$ if the transition is truly continuous. From $S(p,L)$ for the BS2 model obtained from the numerical simulations with $L = 512$–4096 and the relation $p_{\max}(L) = p_c + bL^{-1/\nu}$, where $p_{\max}(L)$ is the p at which $S(p,L)$ is maximal [14], we obtain $\nu = 1.00(1)$ and $p_c = 0.5270(1)$ for the BS2 model. From the relation for the maximal value of $S(p,L)$, $S_{\max} \sim L^{\gamma/\nu}$ [14] and the data obtained for S we obtain $\gamma/\nu = 1.90(2)$. Thus $S(p,L)$ for the BS2 model satisfies the FSS ansatz [14]

$$S(p,L) = L^{\gamma/\nu} f((p - p_c)L^{1/\nu}) \qquad (2)$$

very well with $\nu = 1.00(1)$ and $p_c = 0.5270(1)$, as shown in Fig. 3(a). $P_\infty(p,L)$ for the BS2 model satisfies the FSS ansatz [14]

$$P_\infty(p,L) = L^{-\beta/\nu} g((p - p_c)L^{1/\nu}) \qquad (3)$$

very well with $\beta/\nu = 0.04(1)$ [see Fig. 3(b)]. Since γ and β are related to τ and σ as $\gamma = (3 - \tau)/\sigma$ and $\beta = (\tau - 2)/\sigma$ [14], ν, γ, and β obtained from the FSS ansatz are consistent with τ and σ obtained from Eq. (1) for the BS2

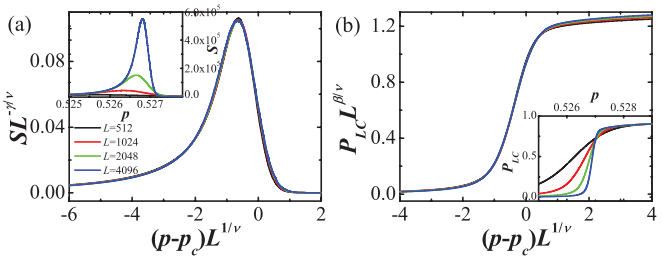

FIG. 3. (Color online) Scaling plots of (a) $S(p,L)$ and (b) $P_{LC}(p,L)$ for the BS2 model. The insets shows the original data for $S(p,L)$ in (a) and $P_{LC}(p,L)$ in (b).

TABLE I. Summary of the transition natures for the six possible models.

Model	Type	When a is an internal bond	Transition nature	Duality or universality
SP	site product	not applicable	discontinuous	dual to BP1
BP1	bond product	$\xi_a = s_{a1} \times 1$	discontinuous	dual to SP
SS	site sum	not applicable	discontinuous	dual to BS1
BS1	bond sum	$\sigma_a = s_{a1} + 0$	discontinuous	dual to SS
BP2	bond product	$\xi_a = s_{a1} \times s_{a1}$	continuous	same universality class as BS2
BS2	bond sum	$\sigma_a = s_{a1} + s_{a1}$	continuous	same universality class as BP2

model. The obtained exponents are $\tau = 2.02(1), \sigma = 0.51(1)$, $\nu = 1.00(1), \gamma = 1.90(2)$, and $\beta = 0.04(1)$. For the BP2 model we obtain exponents and scaling relations identical to those for the BS2 model.

V. BEHAVIOR OF THE SECOND LARGEST CLUSTER

The discontinuous transition is characterized by the existence of metastable states. The metastable states in percolation transition generally originate from the coexistence of multiple macroscopic clusters [19–21]. To study multiple macroscopic clusters, we now focus on $P_2(p,L)$, for which a randomly chosen occupied site belongs to the second largest cluster. If the transition is discontinuous, then P_2 approaches a nonzero value at $p \simeq p_c$ in the limit $L \to \infty$. On the other hand, when the transition is continuous, the largest cluster grows by gradually adding small clusters, and P_2 at $p \simeq p_c$ should decrease to zero in the thermodynamic limit. For the sake of comparison, we first measure P_2 for ordinary bond percolation (OBP). As shown in Fig. 4(a), p at which P_2 is maximal, $p_{2\max}$, approaches the known value of $p_c = 1/2$ as L increases. Furthermore the maximal value $P_2(p_{2\max})$ decreases as L increases. These behaviors of $P_2(p,L)$ clearly show the absence of multiple macroscopic clusters in OBP for $p \simeq p_c$ in the limit $L \to \infty$, and the transition becomes continuous.

In Fig. 4(b) we show $P_2(p,L)$ for the SP and BP1 models. Here we also see the bond-site duality for the BP1 and SP models. Like P_2 in OBP, $p_{2\max}$ for the BP1 and SP models approaches the estimated p_c from Figs. 1(a) and 1(b) as L increases. However, in contrast to OBP, $P_2(p_{2\max})$ for both models does not decrease and remains at a nearly constant value as L increases. This behavior clearly shows that P_2 for both models does not vanish at $p \simeq p_c$ in the limit $L \to \infty$, which provides strong evidence for the discontinuous transition.

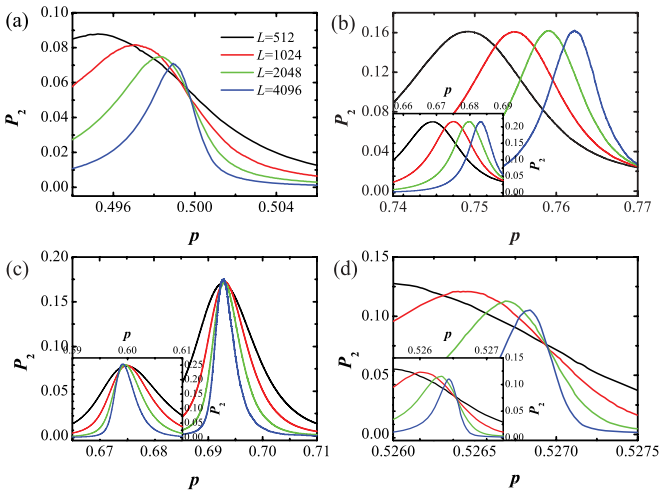

FIG. 4. (Color online) (a) Plot of $P_2(p,L)$ against p (a) for ordinary bond percolation, (b) for the SP model (inset: the BP1 model), (c) for the SS model (inset: the BS1 model), and (d) for the BS2 model (inset: the BP2 model).

In Fig. 4(c) we display $P_2(p,L)$ for the BS1 and SS models. As can be seen from Fig. 4(c), there also exists bond-site duality for the BS1 and SS models. As shown in Fig. 4(c), P_2 for both the BS1 and SS models manifests anomalous behavior. Unlike OBP, $p_{2\max}$ for both models hardly varies as L increases and is very close to the estimated p_c from Figs. 1(c) and 1(d) regardless of L. Furthermore $P_2(p_{2\max})$ for the SS model remains at a nearly constant value or increases slightly as L increases. These results for the BS1 and SS models physically mean that there exists a stable macroscopic second largest cluster even at p_c in the thermodynamic limit, and the transition should be discontinuous.

In contrast, P_2 for the BP2 and BS2 models in Fig. 4(d) is physically very similar to that for OBP in Fig. 4(a). $p_{2\max}$ for the BP2 and BS2 models approaches the estimated p_c as L increases. $P_2(p_{2\max})$ for the BS2 model decreases as L increases like that for OBP, which indicates that P_2 vanishes at $p \simeq p_c$ in the thermodynamic limit. Thus, the transition becomes continuous for both the BP2 and BS2 models, as expected from the $P_s(p)$ data. The transition natures for the six models are summarized in Table I.

VI. SUMMARY AND DISCUSSIONS

In this paper we exactly defined the explosive lattice percolation models on the square lattice. By studying P_s and P_2 for the models, we observed the bond-site duality in the pair of SP and BP1 models and in the pair of SS and BS1 models. The duality means a discontinuous transition. In contrast two bond models, the BP2 and BS2 models, which relatively suppress the intrabond occupation, undergo continuous transition, which satisfies the normal scaling behavior like Eqs. (1), (2), and (3).

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