Bond-site duality and nature of the explosive-percolation phase transition on a two-dimensional lattice

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To establish the bond-site duality of explosive percolations in two dimensions, the site and bond explosive-percolation models are carefully defined on a square lattice. By studying the cluster distribution function and the behavior of the second largest cluster, it is shown that the duality in which the transition is discontinuous exists for the pairs of the site model and the corresponding bond model which relatively enhances the intrabond occupation. In contrast the intrabond-suppressed models which have no corresponding site models undergo a continuous transition and satisfy the normal scaling ansatz as ordinary percolation.

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I. INTRODUCTION

Recently, the Achlioptas process (AP) [1], which was suggested to show a supposedly first order transition on the complete graph, triggered intensive studies on explosive percolations [2–7]. However subsequent studies have proved that the transition in the AP on the complete graph shows a continuous transition [8]. Therefore the transition in the original AP is physically established to be continuous in the mean-field level or in high dimensions.

Until now studies on the AP have been done mainly on the complete graph. Even though there are some studies on the AP in two dimensions (2D) [3,7,9–12], the transition nature in lower dimensions is still not fully understood. For example, the bond percolation under the AP with a product rule was first argued to show a discontinuous transition [3,9]. However, based on the measurement of the largest cluster distribution, Grassberger et al. [7] argued that the bond percolation with the same product rule in 2D undergoes continuous transition [7]. The site percolation under the AP with a product rule in 2D has been proved to undergo a discontinuous transition based on a detailed analysis of cluster size distribution and hysteresis [11]. In contrast, Bastas et al. argued that the site percolation under the AP with a sum rule in 2D undergoes continuous transition based on the finite-size scaling analysis with relatively small system sizes [12]. Those controversies also show that we cannot simply exclude the possibility that the transition nature of the AP in the mean-field limit is different from that in lower-dimensional systems, such as the well-studied Potts model [13].

Such controversies also imply that there does not exist a bond-site duality among explosive-percolation models on 2D lattices, unlike ordinary percolation [14,15]. Here bond-site duality means that a bond percolation model has the same transition nature or belongs to the same universality class as the corresponding site percolation model except for the properties depending on the details of the models, such as transition probability $p_c$. Moreover, if the transition is truly discontinuous, then determination of critical exponents from finite-size scaling analysis as in Ref. [12] has no physical meaning, as we already addressed in Ref. [11]. Thus, resolving such controversies by establishing the bond-site duality in 2D is theoretically very important and interesting.

The controversies should come from ambiguities in the definitions of explosive-percolation models on lattices. Therefore it is very important to make clear definitions of models with various growth rules for the AP. There can be six kinds of models on a 2D square lattice. Among them, two pairs are bond models. One pair consists of the bond models which physically enhance the occupation of intracluster bonds. The other pair consists of bond models which relatively suppress the occupation of intrabonds.

In site percolation, there cannot be a distinction between interbonds and intrabonds in a cluster because the occupation of any empty site always increases the size of the cluster. So there is no ambiguity in the definition of site models as in bond models. As we shall see, the intrabond-enhanced models and the corresponding site models show the bond-site duality in which the transition is discontinuous. In contrast the intrabond-suppressed models show a continuous transition. Physically, there should be no site model corresponding to such intrabond-suppressed models.

II. MODELS

There are two fundamental percolation models on lattices [14]. One is the site percolation model, and the other is the bond percolation model. In the site percolation model, there is no new site occupation which does not change the size of clusters. Under the AP, two vacant sites $A$ and $B$ are randomly selected. Let $(s_A, s_B)$ be the sizes of $n_A$ ($n_B$) clusters which would be connected by occupying site $A$ ($B$). In the square lattice $n_A (n_B)$ is at most 4. In the site model with a product rule (SP model) site $A$ is occupied if $1 \times \prod_{i=1}^{n_A} s_A < 1 \times \prod_{j=1}^{n_B} s_B$. Otherwise, site $B$ is occupied. Similarly, in the site model with a sum rule (SS model) site $A$ is occupied if $(1 + \sum_{i=1}^{n_A} s_A) < (1 + \sum_{j=1}^{n_B} s_B)$.

We define four bond percolation models under AP. The first two unoccupied bonds $a$ and $b$ are selected randomly. If bond $a$ is an interbond, then it connects two different clusters of sizes $s_{a1}$ and $s_{a2}$. Then under a product rule product $\xi_A$ for bond $a$ is clearly defined as $\xi_A \equiv s_{a1} \times s_{a2}$ without any ambiguity. If
the bond is an intrabond, it internally connects two sites in the same cluster. Then $\xi_a$ can be defined in two different ways. One is $\xi_a \equiv s_{a1} \times 1$ [bond product type 1 (BP1) model]. The other is $\xi_a \equiv s_{a1} + s_{a2}$ [bond product type 2 (BP2) model]. The product $\xi_b$ for bond $b$ is similarly defined. Bond $a$ will be occupied if $\xi_a < \xi_b$. Otherwise, bond $b$ will be occupied. Therefore the two bond product models, BP1 and BP2, come from the ambiguity in defining the product for the selected intrabond. Similarly, we can define two kinds of bond models with a sum rule. The sum $\sigma_a$ for interbond $a$ is defined clearly as $\sigma_a \equiv s_{a1} + s_{a2}$. In contrast $\sigma_a$ for intrabond $a$ can also be defined in two different ways: $\sigma_a \equiv s_{a1} + 0$ [bond sum type 1 (BS1) model] or $\sigma_a \equiv s_{a1} + s_{a1}$ [bond sum type 2 (BS2) model]. Then bond $a$ will be occupied if $\sigma_a < \sigma_b$.

The physical meaning of the models is that type 1 models (BP1 and BS1) relatively enhance the intrabond occupation, whereas type 2 models (BP2 and BS2) suppress the intrabond occupation. Thus if bond-site duality exists, it should be between type 1 bond models and site models. As we shall see, the duality exists for the pair of BP1 and SP models and the pair of BS1 and SS models. BP2 and BS2 have no corresponding site models for the duality.

Until now, only the BP2 model has been studied for the explosive bond percolation model [3,7,9]. In Ref. [9] the BP2 model was argued to show a discontinuous transition, whereas in Ref. [7] the same model was argued to undergo a continuous transition. The BP1, BS1, and BS2 models have not been studied. Both SP and SS have been studied as explosive site percolation models. The SP model [11] has been proved to show a discontinuous transition, whereas the SS model [12] was argued to show a continuous transition based on numerical studies of relatively small systems.

In the following sections, based on more precise simulations with larger system sizes, we will show the existence of the duality between bond and site percolation AP models, as well as the characterization of the transition nature of each model. For this purpose, we use $L = 512–8192$, and the data are averaged over $10^6$ independent runs.

### III. Cluster Size Distribution

To understand the transition nature of the percolation physically, the cluster size distribution should be understood first. The cluster size distribution $P_s(p)$ at an occupation probability $p$ of a bond (or site) is the probability that an occupied site belongs to a cluster which has $s$ sites. It has been shown that $P_s(p)$ provides an excellent method to determine $p_c$, as well as the transition nature for an ordinary percolation (OP) [14] and explosive percolations [5,11]. Thus, we first investigate $P_s(p)$ for each model to obtain $p_c$ and the transition nature physically. When $p = p_c$, $P_s$ for OP decays exponentially as $s$ increases. This means that the probability of finding a large cluster vanishes exponentially. On the other hand, when $p > p_c$, there exists a macroscopically large cluster (LC), and $P_s$ for finite $s$ also decays exponentially with a peak for the LC. At $p = p_c$, it is well known that $P_s$ satisfies a power law, $P_s(p_c) \sim s^{-\delta}$, with $\delta \approx 1.055$ for OP.

In contrast, $P_s$ for the SP model in Fig. 1(a) is completely different from that for OP, as shown in Ref. [11]. When $p < p_c$, $P_s$ for the SP model has a stable hump in the tail as $p \to p_c$.
obtain \( P_r \sim s^\delta \) with \( \delta \simeq 0.95(1) (\langle 1 \rangle) \). For \( p > p_c \), we find that \( P_r \) for finite \( s \) still satisfies the power law with \( \delta < 1 \) (see the bottom inset). These behaviors of \( P_r(p) \) for the BP1 model exactly coincide with that for the SP model, which is known to undergo discontinuous transition \([11]\). The only difference between the BP1 model and the SP model is in the value of \( p_c: p_c \simeq 0.6937(8) \) for the BP1 model and \( p_c \simeq 0.7723(7) \) for the SP model. The coincidence of \( P_r \) between the BP1 model and the SP model physically means that there is a bond-site duality under the AP with the product rule if the bond model enhances the intrabond occupation.

Figures 1(c) and 1(d) show \( P_r \) for the BS1 and SS models. By assuming the relation \( P_r \sim s^\delta \) only for small \( s \) at \( p \simeq p_c \), we obtain \( \delta \simeq 1.5 \) for the BS1 and SS models. However, as depicted in Figs. 1(c) and 1(d), \( P_r \) for the BS1 and SS models seems to substantially deviate from the power law due to a main contribution from the hump distribution for large \( s \). Even for the value of \( s \) at which \( P_r \) for LC starts to become detached from the seemingly power-law-like regime, the hump still exists. The contribution of the hump distribution part to \( \sum_{s=1}^{L^2} P(s) = p \) for site percolation [or \( \sum_{s=1}^{L^2} P(s) = 1 \) for bond percolation] even at \( p \simeq p_c \) is more than 90\% \([17]\). This means that the absence of the singular behavior, \( P_r(p) \sim s^{-\delta} \) for \( s \to \infty \) at \( p_c \), which is the intrinsic property of the continuous transition. Moreover in both BS and SS models the hump distribution for \( p < p_c \) does not depend on the lattice size \( L \) (see the top insets). This anomalous behavior suggests that there exists a type of powder keg \([18]\) in the BS1 and SS models, unlike in the fully connected networks \([6]\). In addition, the existence of the hump distribution even for \( p > p_c \) (see the bottom insets) implies the existence of multiple stable macroscopic clusters for \( p \simeq p_c \) in the BS1 and SS models. Furthermore, \( P_r \) for the BS1 model \([14]\) is nearly identical to \( P_r \) for the SS model \([14]\) as in the case for the BP1 and SP models. The coincidence of \( P_r \) between the BS1 and SS models except for the value of \( p_c \) also implies that there is bond-site duality under the AP with the sum rule if the bond model enhances the intrabond occupation.

In Figs. 1(e) and 1(f) we display \( P_r \) for the BP2 and BS2 models. \( P_r \) for the BP2 and BS2 models is physically different from \( P_r \) for the BP1 and BS1 models. Even though \( P_r \) for the BP2 and BS2 models has a hump in the tail when \( p < p_c \) (top inset), this hump vanishes in the limit \( L \to \infty \), and \( P_r \) at \( p_c \) shows the power-law singularity \( P_r(p) \sim s^{-\delta} \) with \( \delta = 1.02(1) > 1 \). As we increase \( p \) further to \( p > p_c \), \( P_r \) for finite \( s \) decays exponentially as in OP (see the bottom insets for \( p > p_c \)). This \( P_r \) behavior is physically the same as that in OP. As for OP, \( P_r \) or \( n_x(= P_r(s)) \) at \( p_c \) for the percolation with continuous transition satisfies the scaling relation \([14]\)

\[
P_r = s^{-\delta} f(s/L^{1/\nu})[n_x = s^{-\tau} f(s/L^{1/\tau+\nu})],
\]

where \( \delta = \tau - 1 + \nu \) is the correlation length critical exponent. As shown in Figs. 2(a) and 2(b), \( P_r(n_x) \) for both the BP2 and BS2 models satisfies scaling relation (1) with \( \delta = 1.02 (\tau = 2.02) \) and \( \nu = 0.51 \) very well. \( P_r \) in Figs. 1(e), 1(f), 2(a), and 2(b) strongly suggests that both the BS2 and BP2 models show continuous transition. Our estimation of \( p_c \) for the BP2 model is nearly the same as those in Refs. \([7,9]\), even though the two references were contradictory to each other in transition nature.

IV. FINITE-SIZE SCALING ANALYSIS FOR THE BP2 AND BS2 MODELS

From \( \tau \) and \( \sigma \) obtained from Eq. (1), one can calculate the critical exponents \( \beta, \gamma, \) and \( \nu \), which must be identical to the values evaluated from the finite-size scaling (FSS) properties of the average-size \( S(p,L) \) of the finite clusters and the order parameter \( P_{\infty}(p,L) \) if the transition is truly continuous. From \( S(p,L) \) for the BS2 model obtained from the numerical simulations with \( L = 512-4096 \) and the relation \( P_{\max}(L) = p_c + bL^{-1/\nu} \), where \( P_{\max}(L) \) is the \( p \) at which \( S(p,L) \) is maximal \([14]\), we obtain \( \nu = 1.00(1) \) and \( p_c = 0.5270(1) \) for the BS2 model. From the relation for the maximal value of \( S(p,L), S_{\max} \sim L^{\gamma/\nu} \) \([14]\) and the data obtained for \( S(p,L) \) we obtain \( \gamma/\nu = 1.90(2) \). Thus \( S(p,L) \) for the BS2 model satisfies the FSS ansatz \([14]\)

\[
S(p,L) = L^{\gamma/\nu} f((p - p_c)L^{1/\nu})
\]

very well with \( \nu = 1.00(1) \) and \( p_c = 0.5270(1) \), as shown in Fig. 3(a). \( P_{\infty}(p,L) \) for the BS2 model satisfies the FSS ansatz \([14]\)

\[
P_{\infty}(p,L) = L^{-\beta/\nu} g((p - p_c)L^{1/\nu})
\]

very well with \( \beta/\nu = 0.04(1) \) [see Fig. 3(b)]. Since \( \gamma \) and \( \beta \) are related to \( \tau \) and \( \sigma \) as \( \gamma = (3 - \tau)/\sigma \) and \( \beta = (\tau - 2)/\sigma \) \([14]\), \( \nu, \gamma, \) and \( \beta \) obtained from the FSS ansatz are consistent with \( \tau \) and \( \sigma \) obtained from Eq. (1) for the BS2 model.
model. The obtained exponents are $\tau = 2.02(1), \sigma = 0.51(1),
\nu = 1.00(1), \gamma = 1.90(2)$, and $\beta = 0.04(1)$. For the BP2
model we obtain exponents and scaling relations identical to
those for the BS2 model.

V. BEHAVIOR OF THE SECOND LARGEST CLUSTER

The discontinuous transition is characterized by the exist-
ence of metastable states. The metastable states in percolation
transition generally originate from the coexistence of multiple
macroscopic clusters [19–21]. To study multiple macroscopic
clusters, we now focus on

\[
P_2(p) = \frac{\alpha_{\text{max}}}{\alpha_{\text{est}}}
\]

for both the BS1 and SS models. Unlike OBP, $P_2(p_{\text{max}})$
for both models hardly varies as $L$ increases and is very close to
the estimated $p_c$, from Figs. 1(c) and 1(d) regardless of $L$. Furthermore $P_2(p_{\text{max}})$
for the SS model remains at a nearly constant value or increases slightly as $L$
increases. These results for the BS1 and SS models physically
mean that there exists a stable macroscopic second largest
cluster even at $p_c$ in the thermodynamic limit, and the transition
should be discontinuous.

In contrast, $P_2$ for the BP2 and BS2 models in Fig. 4(d)
is physically very similar to that for OBP in Fig. 4(a). $P_2(p_{\text{max}})$
for the BP2 and BS2 models approaches the estimated $p_c$ as $L$
increases. $P_2(p_{\text{max}})$ for the BS2 model decreases as $L$
increases like that for OBP, which indicates that $P_2$ vanishes at $p \simeq p_c$.
In the thermodynamic limit, the transition
should be discontinuous.

In Fig. 4(c) we display $P_2(p,L)$ for the BS1 and SS models.
As can be seen from Fig. 4(c), there also exists bond-site
duality for the BS1 and SS models. As shown in Fig. 4(c), $P_2$
for both the BS1 and SS models manifests anomalous behavior.
Unlike OBP, $P_2(p_{\text{max}})$ for both models hardly varies as $L$
increases and is very close to the estimated $p_c$ from Figs. 1(c) and
1(d) regardless of $L$. Furthermore $P_2(p_{\text{max}})$ for the SS model
remains at a nearly constant value or increases slightly as $L$
increases. These results for the BS1 and SS models physically
mean that there exists a stable macroscopic second largest
cluster even at $p_c$ in the thermodynamic limit, and the transition
should be discontinuous.

In contrast, $P_2$ for the BP2 and BS2 models in Fig. 4(d)
is physically very similar to that for OBP in Fig. 4(a). $P_2(p_{\text{max}})$
for the BP2 and BS2 models approaches the estimated $p_c$ as $L$
increases. $P_2(p_{\text{max}})$ for the BS2 model decreases as $L$
increases like that for OBP, which indicates that $P_2$ vanishes at $p \simeq p_c$.

VI. SUMMARY AND DISCUSSIONS

In this paper we exactly defined the explosive lattice
percolation models on the square lattice. By studying $P_1$ and $P_2$
for the models, we observed the bond-site duality in the pair of
SP and BP1 models and in the pair of SS and BS1 models. The
duality means a discontinuous transition. In contrast two bond
models, the BP2 and BS2 models, which relatively suppress the
intrabond occupation, undergo continuous transition, which
satisfies the normal scaling behavior like Eqs. (1), (2), and (3).

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[16] In this study by using larger systems and averaging over more samples we obtain more accurate values for $p_c$ and $\delta$ compared with those in Ref. [11].
[17] Even though the height of humps at $s = s^*$ in $P(s)$ decreases as a power law, $P(s^*; L) \sim L^{-a}$ with $a = 1.3(1)$, $s^*$ also increases as $s^*(L) \sim L^b$ with $b = 1.4(1)$ as $L$ increases. Therefore, $\sum_{s \sim s^*} P(s) \Delta s|_{s \sim s^*}$ remains a constant within the estimated error.