Explosive percolation on the Bethe lattice

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Based on self-consistent equations of the order parameter P_∞ and the mean cluster size S, we develop a self-consistent simulation method for arbitrary percolation on the Bethe lattice (infinite homogeneous Cayley tree). By applying the self-consistent simulation to well-known percolation models, random bond percolation, and bootstrap percolation, we obtain prototype functions for continuous and discontinuous phase transitions. By comparing key functions obtained from self-consistent simulations for Achlioptas models with a product rule and a sum rule to the prototype functions, we show that the percolation transition of Achlioptas models on the Bethe lattice is continuous regardless of details of growth rules.

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I. INTRODUCTION

Since Achlioptas *et al*. [1] suggested an explosive percolation model, there have been intensive studies on explosive percolations [2–12]. The Achlioptas process was originally argued to show discontinuous phase transition on the complete graph by suppressing growth of large clusters [1]. Subsequent studies on variants of explosive percolation models on networks and lattices also argued to show the discontinuous transition [2–8]. However, Riordan and Warnke [12] have analytically showed that the phase transition in Achlioptas model [1] on the complete graph is continuous by use of the arbitrary connectivity of the complete graph. Furthermore, several studies also showed continuous transitions for explosive percolation models on the complete graph [9–11]. In contrast, explosive percolation on the complete graph might show a discontinuous transition by controlling some of the cluster growth rules [13]. Therefore, the transition nature of explosive percolation on the complete graph is not perfectly and physically understood. The physics on the complete graph should satisfy mean-field theory. In this sense the mean-field theory for explosive percolation still needs to be checked by use of some other medium instead of a complete graph.

The Bethe lattice (infinite homogeneous Cayley tree) is physically a very important substrate or medium on which mean-field theories for various physical models become exact [14]. The analytic treatments of magnetic models [15], percolation [14,16], localization [14], and diffusion [17] on the Bethe lattice give important physical insights to subsequent developments of the corresponding research fields. One of the theoretical merits of the Bethe lattice is that one can set up exact self-consistent equations. In this paper, by use of the exact self-consistent equations, we develop a self-consistent simulation method for the arbitrary percolation process on the Bethe lattice. From the self-consistent simulation method, we precisely calculate the order parameter P_∞ and the mean cluster size S of the Achlioptas model (AM) [1] with a product rule or a sum rule. The obtained P_∞ and S can clarify the transition nature of AM in the infinite dimension exactly. Furthermore, unlike the complete graph, the Bethe lattice has only local bonds. Since there have been some papers in which explosive percolation models on lattices with local bond connections show discontinuous transition [2,4,18], it is physically important to study explosive percolation models on the Bethe lattice or in the mean-field level with local connections. As we shall see, the transition of AM on the Bethe lattice is continuous regardless of the sum rule or the product rule. This result physically means that AM with local connections in the mean-field level shows continuous transition regardless of growth rules.

II. SELF-CONSISTENT SIMULATION

Let us first set up self-consistent equations for arbitrary percolation on the Bethe lattice. To set up self-consistent equations on the Bethe lattice in Fig. 1, one starts with a center site (or origin **O**) having z bonds. First consider a part of the Bethe lattice with n generations from **O**, which have total $N = 1 + z(k^n - 1)/(k - 1)$ sites, where $k = z - 1$. To make a complete Bethe lattice, one should add an infinite branch to each of zk^{n-1} edge sites. To calculate the order parameter P_∞ of percolation, which is defined by the probability of **O** to belong to an infinite cluster at the occupation probability p of a bond (a site), we need to know the probability A with which an occupied edge site connected to an infinite cluster. Let $P_{n\infty}(p,A)$ be P_∞ which is calculated from a Bethe lattice with the n generations from **O** and zk^{n-1} infinite branches. Then the self-consistent equation for P_∞ is

$$P_\infty = P_{n\infty}(p,A) = P_{n'\infty}(p,A) \tag{1}$$

for any combination of $\{n,n'\}$. For the random site percolation Eq. (1) with the combination $\{n = 1,n' = 2\}$ gives $R = 1 - p + pR^k$ with $R \equiv 1 - pA$, which analytically reproduces the mean-field transition of random percolation [14,16]. Let us define $P_{nst}(p,A,S_b)$ as the probability that a cluster including **O** with s sites and t edge sites occurs within the n-generation tree. Then

$$P_{n\infty}(p,A) = 1 - \sum_t (1-A)^t \sum_s P_{nst}(p,A,S_b), \tag{2}$$

where S_b is the average size of the finite cluster connected to an edge site of the n-generation tree in Fig. 1. The self-consistent equation for the average size S of the finite clusters including

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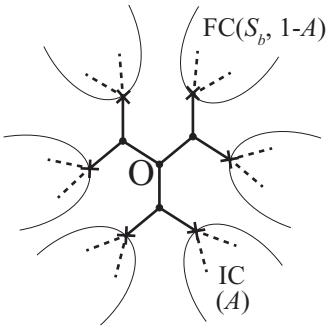

FIG. 1. Schematic diagram for a Bethe lattice with $z = 3$. The center part of the Bethe lattice is a two-generation Cayley tree with six edge sites denoted by "X." Each edge site is connected to an infinite branch. Each edge site is connected to an infinite cluster (IC) with the probability A or to a finite cluster (FC) of average size S_b with the probability $1 - A$.

O can also be written as

$$S = S_n(p,A,S_b) = S_{n'}(p,A,S_b), \tag{3}$$

where

$$S_n(p,A,S_b) = \frac{\sum_{s,t} P_{nst}(p,A,S_b)[s + tS_b](1-A)^t}{1 - P_\infty}. \tag{4}$$

If one cannot calculate $P_{nst}(p,A,S_b)$ analytically or if one needs to know A and S_b *a priori* to occupy a new bond (or site), one should estimate $P_{nst}(p,A,S_b)$ by solving self-consistent equations (1) and (3) indirectly. One of the indirect methods is a simulation method. We develop a simulation method to solve self-consistent equations, which we call the self-consistent simulation. In the self-consistent simulation, $P_{nst}(p,A,S_b)$ is estimated by the relation $P_{nst}(p,A,S_b) = N_{nst}(p,A,S_b)/N_{\text{cluster}}$, where $N_{nst}(p,A,S_b)$ is the number of clusters including **O** with s sites and t edge sites within the n-generation tree that occurred in simulations. Of course, N_{cluster} is the total number of clusters which includes **O** within the n-generation tree that occurred in the same simulation runs. In the simulation both $P_{nst}(p,A,S_b)$ and $P_{n'st}(p,A,S_b)$ are estimated simultaneously using the Bethe lattice with the n-generation tree if $n > n'$.

Since we don't know A and S_b *a priori*, the iteration processes are needed in the self-consistent simulations. From initial values of A and S_b, the final or saturated values of A and S_b are obtained by the iteration of a unit simulation process at the given p. The unit simulation process consists of the following two steps. (I) In the (i)th iteration let the values of A and S_b become $A^{(i)}$ and $S_b^{(i)}$. Using $A^{(i)}$ and $S_b^{(i)}$, P_{nst} and $P_{n'st}$ are estimated by the relation $P_{nst} = N_{nst}/N_{\text{cluster}}$. (II) From the estimated P_{nst} and $P_{n'st}$, $P_{n\infty}(p,A^{(i+1)})$ and $P_{n'\infty}(p,A^{(i+1)})$ are calculated by the relation $P_{n\infty}(p,A^{(i+1)}) = 1 - \sum_t (1 - A^{(i+1)})^t \sum_s P_{nst}(p,A^{(i)},S_b^{(i)})$. By solving equation $P_{n\infty}(p,A^{(i+1)}) = P_{n'\infty}(p,A^{(i+1)})$, $A^{(i+1)}$ for the $(i+1)$th iteration is obtained. $S_b^{(i+1)}$ is similarly obtained based on Eqs. (3) and (4). In the unit simulation process to get $A^{(i)}$ and $S_b^{(i)}$, $P_{nst}(p,A^{(i)},S_b^{(i)})$ and other relevant terms are estimated by averaging over at least 10^6 simulation runs. Such a unit process is repeated until A and S reach the saturation values or both $A^{(i+1)} = A^{(i)}$ and $S_b^{(i)} = S_b^{(i+1)}$ are satisfied. Using the saturated values of A and S_b, P_∞ and S are estimated from Eqs. (2) and (4) at the given p.

III. RANDOM BOND PERCOLATION AND BOOTSTRAP SITE PERCOLATION

The phase transition of random percolation on the Bethe lattice is well known to be continuous [16]. We have applied our self-consistent simulation to random bond percolation on the Bethe lattice with $z = 4$. The results for $z = 4$, $n = 13$, and $n' = 5$ are displayed in Figs. 2(a) and 2(b). In Fig. 2(a) we display the simulation results for

$$f_p(A) \equiv P_{n'\infty}(p,A) - P_{n\infty}(p,A). \tag{5}$$

From Eq. (1), A^* which satisfies $f_p(A^*) = 0$ is the real physical value at a given p. For $p < p_c$ there is only a trivial solution $A^* = 0$ as shown in Fig. 2(a). Increasing

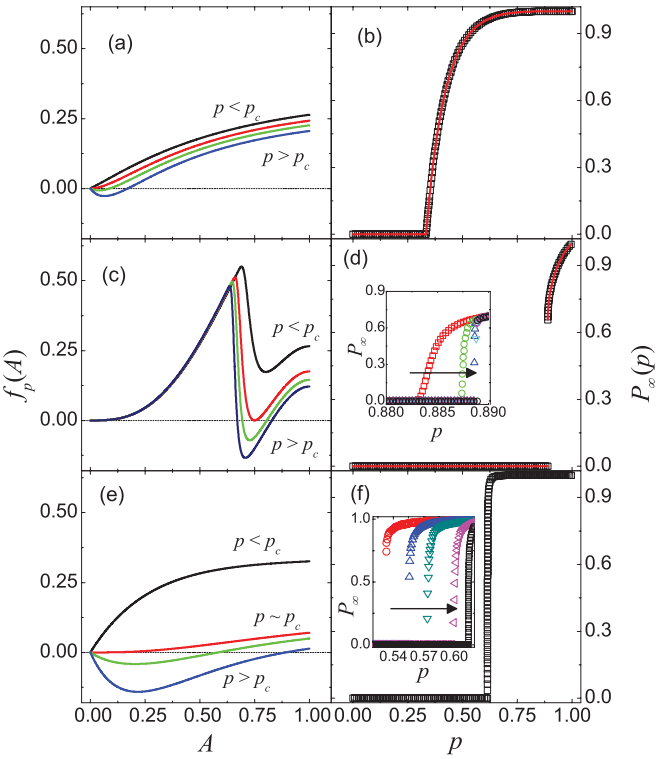

FIG. 2. (Color online) Self-consistent simulation results of $f_p(A)$ and P_∞ for random bond percolation, bootstrap site percolation, and AM on the Bethe lattice with $z = 4$. (a) $f_p(A)$ of random bond percolation for $p = 0.31333(<p_c)$, $p = 0.33333(\simeq p_c = 1/3)$, $p = 0.34333(>p_c)$, and $p = 0.35333(>p_c)$. (b) Plot of P_∞ for the random percolation against p. The line means the analytic result. (c) $f_p(A)$ of bootstrap site percolation for $p = 0.87888(<p_c)$, $p = 0.88888(\simeq p_c = 8/9)$, $p = 0.89388(>p_c)$, and $p = 0.89888(>p_c)$. (d) Plot of P_∞ for the bootstrap percolation against p. The line means the analytic result. Inset shows the evolution of results for iteration processes. The arrow $\rightarrow$ denotes the direction from the result for the earlier iteration process to that for the later iteration process. (e) $f_p(A)$ of AM with a product rule for $p = 0.60575(<p_c)$, $p = 0.61575(\simeq p_c)$, $p = 0.61775(>p_c)$, and $p = 0.62575(>p_c)$. (f) Plot of P_∞ for AM with the product rule against p. Inset shows the evolution of results for iteration processes. The arrow $\rightarrow$ means the same thing as in (d).

p from p_c, the nontrivial solution $A^* > 0$ continuously increases from zero. This continuous increase makes $P_\infty(p)$ increase continuously as in Fig. 2(b). The simulation result for $P_\infty(p)$ exactly coincides with the analytic result $P_\infty = 1 - [1/2 - \sqrt{(4-3p)/4p}]^4$ for $p > p_c(=1/k = 1/3)$ as shown in Fig. 2(b). Therefore, $f_p(A)$ which behaves like in Fig. 2(a) is a prototype function for the continuous transition.

Bootstrap site percolation on the Bethe lattice [19] is analytically known to show the discontinuous transition. In bootstrap site percolation P_∞ is the probability for an occupied site to be a site of the infinite m cluster. Here the m cluster means the cluster in which every occupied site has at least m occupied nearest neighbors. For $m \geqslant 3$ the phase transition of bootstrap percolation on the Bethe lattice is discontinuous [19]. By using our self-consistent simulation we have obtained $f_p(A)$ and P_∞ for bootstrap percolation with $m = 3$ on the Bethe lattice with $z = 4$. The results for bootstrap percolation with $z = 4$, $m = 3$, $n = 13$, and $n' = 1$ are shown in Figs. 2(c) and 2(d). As shown in Fig. 2(c), $f_p(A) = 0$ for $p < p_c$ has only the trivial solution $A^* = 0$ as in random percolation. In contrast to random percolation the nontrivial solution of $f_p(A) = 0$ for $p > p_c$ comes from the peculiar behavior of $f_p(A)$, which reminds us of the thermodynamic instability in the thermal mean-field first order transition, which makes the sudden jump of A^* from zero at $p = p_c = 8/9$. The jump of A^* causes the discontinuous increase of P_∞ as shown in Fig. 2(d), which is exactly the same as the analytic result, $P_\infty = p^4A^3(4 - 3pA)$ with $A = 0$ for $p < p_c$ and $A = [3p + 3\sqrt{p - p_c}]/4p^2$ for $p > p_c$ Therefore, $f_p(A)$ like in Fig. 2(c) is a prototype function for the discontinuous transition. Furthermore, as shown in the inset of Fig. 2(d), the iteration processes in self-consistent simulation for the bootstrap site percolation with the initial value $A = 1$ drives $P_\infty(p)$ from the continuous increase to the final discontinuous jump at $p = p_c$, which is also a typical behavior of the discontinuous transition on the Bethe lattice.

IV. ACHLIOPTAS MODEL

We now focus on the Achlioptas model (AM) [1]. To occupy a bond in the AM, two bonds a and b are randomly chosen. Let $\{S_{a1}, S_{a2}\}$ be the sizes of two clusters which would be connected by occupying bond a and $\{S_{b1}, S_{b2}\}$ be the sizes of two clusters for the bond b. Under a product rule, the bond a is made to be occupied and b is discarded if $\prod_{j=1}^{2} S_{aj} < \prod_{j=1}^{2} S_{bj}$. Otherwise, the bond b is made to be occupied. AM with a sum rule is the same as that with the product rule except for the condition $\sum_{j=1}^{2} S_{aj} < \sum_{j=1}^{2} S_{bj}$. Note that an arbitrary edge site is connected to an infinite cluster with the probability A depicted as in Fig. 1. Therefore, S_{a1}, for example, is calculated as

$$S_{a1} = s_{a1} + \infty I_{a1} + (t_{a1} - I_{a1})S_b, \tag{6}$$

where s_{a1} is the number of sites within the n-generation tree in the cluster $a1$, t_{a1} is the number of edge sites, and I_{a1} is the number of edge sites which are connected to the infinite cluster. Of course, I_{a1} depends on A. Therefore, $P_{nst}(p, A, S_b)[= N_{nst}(p, A, S_b)/N_{\text{cluster}}]$ depends *a priori* on A and S_b, so that the iteration is essential in the self-consistent simulation for AM. For the self-consistent simulation of AM, one should be careful to choose $n'(<n)$ for a given n. If n' is too small, the clusters within the n'-generation tree cannot have physical properties of AM enough to give physically plausible solutions for self-consistent equations (1) and (3). If n' is very close to n, $P_{n\infty}(p, A)$ is numerically not so much distinct from $P_{n'\infty}(p, A)$ and the self-consistent equation (1) hardly gives the physically right solution. From the simulations with various sets of $\{n, n'\}$ it is confirmed that a suitable choice of n' should be in the interval $n/3 < n' < n/2$. $f_p(A)$ and P_∞ for AM with the product rule from the self-consistent simulation with $z = 4$, $n = 14$, and $n' = 5$ are displayed in Figs. 2(e) and 2(f). The results for AM with the sum rule are nearly the same as those in Figs. 2(e) and 2(f) except that p_c for the sum rule is slightly smaller than p_c for the product rule. As can be seen from Fig. 2(e), $f_p(A) = 0$ has only trivial solution $A^* = 0$ for $p < p_c$. Increasing p from p_c, the nontrivial solution $A^*(>0)$ continuously increases from zero. A^* for AM increases very rapidly compared to A^* for the random percolation. Except for this rapid increase, $f_p(A)$ for AM is nearly the same as that for the random bond percolation in Fig. 2(a), which is the prototype function for the continuous phase transition. We cannot find any symptom that $f_p(A)$ for AM behaves like $f_p(A)$ for the bootstrap percolation in Fig. 2(c). The continuous increase of A^* makes the order parameter $P_\infty(p)$ for AM increase continuously as in Fig. 2(f). Moreover, as can be seen from the inset of Fig. 2(f) the iteration processes for AM drive $P_\infty(p)$ from the discontinuous jump to the final continuous increase at $p = p_c$, contrary to those in the inset of Fig. 2(d).

In the self-consistent simulation on the Bethe lattice with the n-generation tree, we cannot obtain P_∞ against all the continuous $p \in (0,1)$. Instead, in the simulation, p is increased discretely by $\Delta p = 1/(N-1)$, where N is the total number of sites in the n-generation tree. Therefore, in the simulation the lowest nonzero $P_\infty^{\,1}$ occurs at $p = p_c'$ very close to the true p_c with $0 < (p_c' - p_c) \leqslant \Delta p$. In Fig. 3 we display the dependence of $P_\infty^{\,1}$ on Δp of AM with $\{n, n'\} = \{7,4\},\{9,4\},\{11,5\},\{13,5\},\{14,5\}$. As can be seen from Fig. 3, $P_\infty^{\,1}$ for both AMs with the product rule and the sum rule decreases monotonically to zero as Δp decreases to zero. This result for $P_\infty^{\,1}$ also supports the fact that the phase transition nature of AM on the Bethe lattice is continuous.

From the self-consistent simulation based on Eq. (3), the average size S of the finite clusters is obtained for the random bond percolation and AM with the product rule as in Fig. 4. S for AM with the sum rule shows nearly the same behavior as Fig. 4(b). It is confirmed that S for the random percolation from the simulation is nearly identical to the analytic result

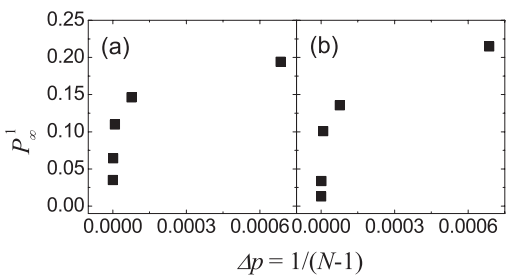

FIG. 3. $P_\infty^{\,1}$ against $\Delta p[\equiv 1/(N-1)]$ for $\{n, n'\} = \{7,4\}$, $\{9,4\},\{11,5\},\{13,5\},\{14,5\}$. (a) AM with a product rule and (b) with a sum rule.

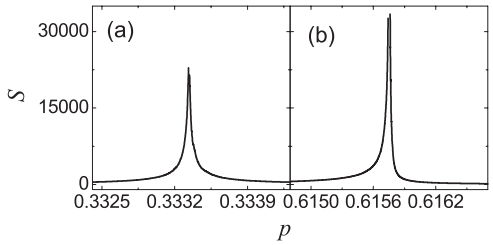

FIG. 4. S on the Bethe lattice with $z = 4$. (a) Plot of S for the random bond percolation against p. The data are obtained from the self-consistent simulation with $n = 13$ and $n' = 5$. (b) Plot of S for AM with the product rule against p. The data are obtained from the simulation with $n = 14$ and $n' = 5$.

$S = 1 + 4p(1 - A)/(1 - 3p + 2pA)$, where $A = 0$ for $p < p_c$ and $A = [3 - \sqrt{(4 - 3p)/p}]/2$ for $p > p_c$. Even though S for AM diverges more rapidly than S for the random percolation, S for both models diverges as $S(p) \simeq |p - p_c|^{-\gamma}$ with the susceptibility exponent $\gamma = 1.00(1)$. The result for S of AM also supports that the transition in AM is continuous on the Bethe lattice.

Since the transition in AM is continuous, the order parameter exponent β is calculated by fitting the relation $P_\infty \simeq (p - p_c)^\beta$ to the data for $p > p_c$ but very close to p_c in Fig. 2(f). Since $f_p(A)$ for $p \simeq p_c$ in Fig. 2(e) increases very slowly as A increases from zero, P_∞ increases very rapidly and the exponent β is expected to be very small. From the best fit the obtained exponent is $\beta = 0.05(5)$. The data even fits very well to $P_\infty \simeq |\ln(p - p_c)|^{-\chi}$ with $\chi = 3.4(1)$. This result means that the exponent β on the Bethe lattice is very small and nearly identical to those obtained on the complete graph [4,7,10].

V. SUMMARY

In summary, we show that the phase transition in AM on the Bethe lattice is continuous from the self-consistent simulations on the Bethe lattice. For this $f_p(A)$ for AM is shown to be physically identical to that for random bond percolation. We also show that $P_\infty{}^1$ for AM decrease to zero as $\Delta p = 1/(N - 1)$ goes to zero. S for AM satisfies $S(p) \simeq |p - p_c|^{-\gamma}$ with $\gamma = 1$. The exponent β is also shown to be very small or $\beta \simeq 0.05$.

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