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Explosive site percolation with a product rule

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We study the site percolation under Achlioptas process with a product rule in a two-dimensional square lattice. From the measurement of the cluster size distribution P_s, we find that P_s has a very robust power-law regime followed by a stable hump near the transition threshold. Based on the careful analysis on the P_s distribution, we show that the transition should be discontinuous. The existence of the hysteresis loop in order parameter also verifies that the transition is discontinuous in two dimensions. Moreover, we also show that the transition nature from the product rule is not the same as that from a sum rule in two dimensions.

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The percolation transition describing the emergence of large-scale connectivity in lattice systems or complex networks has been extensively studied in statistical mechanics and related fields due to its possible applications to various phenomena such as sol-gel transition and polymerization, resistor networks, and epidemic spreading [1]. When the occupation probability of node (site) is lower than a certain threshold p_c, all the clusters are microscopic. As the occupation probability increases, the macroscopically connected cluster emerges. Such transition in the ordinary percolation is continuous [1].

On the other hand, there have been several attempts to find a percolation model which undergoes a discontinuous transition. The discontinuous percolation transition can be found in the modeling of magnetic systems with significant competition between exchange and crystal-field interactions [2,3]. A similar phenomenon has been found in financial systems [4], in which two equally probable phases exist. Other examples of the discontinuous transition in percolation are the formation of infinite cluster under a central force [5] and the cascade of failures in interdependent networks [6].

Recently, Achlioptas *et al*. [7] suggested a simple process in which the growth of large clusters is systematically suppressed and the process is usually called an Achlioptas process (AP). Based on the analysis of transition interval it was argued that the percolation transition under AP is *explosive* and discontinuous. Several variants of models have been investigated to understand the general properties and conditions which cause such nontrivial discontinuous transition [8–11]. Some examples of such nontrivial transition has been found in nanotube based systems [11], protein homology networks [12], and community formation [13].

However, more recent studies on the percolation transition under APs reveal several evidences which strongly suggest that the transition can be continuous. For example, da Costa *et al*. [14] argued that the transition in the complete graph (CG) is continuous, even though the order parameter exponent is very small ($\beta \simeq 0.056$). From the measurement of the cluster size distribution Lee *et al*. [15] argued that the transition in CG is continuous. Riordan and Warnke [16] also analytically showed that the product rule on bond percolation leads to a continuous transition in CG. Grassberger *et al*. [17] also argued that the transition, even in the low-dimensional systems, can be continuous based on a measurement of the order parameter distribution.

Since most of the studies on the criticality of the AP process are restricted to the infinite dimensional systems, it is still not clear whether or not AP also produces a continuous transition in lower dimensional systems. For example, in the bond percolation under AP in a two-dimensional (2D) square lattice, the product rule was argued to produce a discontinuous transition based on a finite-size scaling [8,18]. In contrast Grassberger *et al*. [17] argued that transition of the 2D AP bond percolation is still continuous. We therefore cannot exclude the possibility that the transition nature of the AP in the mean-field limit can be different from that in lower-dimensional systems, like the Potts model [19]. Moreover, based on the measurement of hysteresis [20], a sum rule for the 2D site percolation possibly makes the transition continuous in the thermodynamic limit. This indicates that under the AP-like processes the bond percolation and site percolation may have different transition natures in the 2D lattice. In the ordinary percolation, bond and site percolations are known to belong to the same universality class [1]. In contrast, the results in Refs. [8,18,20] show the possibility that under AP the bond percolation with a product rule and the site percolation with a sum rule do not belong to the same universality class in two dimensions. Therefore validity for the duality between bonds and site percolation under AP in low dimension is one of the very important open questions. And it is also theoretically important and interesting to investigate whether or not in a low-dimensional system the product rule and the sum rule belong to the same universality class. In order to achieve this purpose, we investigate the site percolation under AP with a product rule and show that AP with the product rule produces a clear discontinuous transition in a 2D lattice. For this we carefully analyze the cluster size distribution and hysteresis.

AP in 2D site percolation is defined as follows: (I) We select two sites α and β at random. (II) Let $\{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}\}$ ($\{s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_m}\}$) be the sizes of clusters which form into a new big cluster with the size $\sum_{k=1}^{n} s_{\alpha_k} + 1$ ($\sum_{k=1}^{m} s_{\beta_k} + 1$) by occupying the site α (β). Here the cluster size is defined by the number of sites in the cluster. Then calculate the products

$$\pi_\alpha = \prod_{i=1}^{n} s_{\alpha_i} \quad \text{and} \quad \pi_\beta = \prod_{j=1}^{m} s_{\beta_j}. \tag{1}$$

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(III) If $\pi_\alpha \leqslant \pi_\beta$ ($\pi_\alpha \geqslant \pi_\beta$) then site β (α) remains vacant. This rule is generally called the product rule (PR). Processes (II) and (III) prefer the connection between small clusters, which suppresses the growth of a large cluster. If the product in Eq. (1) is replaced by summation, then the rule is called a sum rule. Since we use a 2D square lattice, $n(m)$ in Eq. (1) is at most 4, while n (m) for bond percolation is always 2. Furthermore, the increase of the dimension leads to an increase in n (m), thus the merging of small clusters is not always favored in higher-dimensional systems.

To understand the percolation transition physically to a deeper level, the properties of the cluster size distribution should be the first one to understand [1,15,21]. The cluster size distribution $P_s(p)$ at a site occupation probability p is normally defined by the probability that a randomly selected site belongs to a cluster which has s sites (s cluster). For ordinary percolation, it is well known that $P_s(p_c)$ satisfies a power-law relation [1],

$$P_s(p_c) \sim s^{-\delta}, \tag{2}$$

with $\delta \simeq 1.05$. Since, as we shall show, the percolation properties under AP depend on the history of how the clusters are grown, we measure $P_s(p)$ by filling sites from the vacant lattice or increasing p. In Fig. 1 $P_s(p)$'s for 2D site percolation under AP with product rule (2DSAP) are displayed. P_s in Fig. 1 shows an anomalously unique behavior compared to that of the ordinary percolation (OP) [1] and that of the AP percolation on the complete graph (APCG), which was argued to undergo the continuous phase transition [15].

When $p < p_c$, P_s for 2DSAP has a hump in the tail as p approaches to p_c [21]. In this regime, P_s does not depend on L or $N(=L \times L)$ as shown in Fig. 1(a). In OP, P_s normally decays exponentially as s gets larger in this regime. For the detailed comparison to those of OP and APCG, let us call s at which the hump is maximal s_H. In OP we cannot identify s_H. In APCG s_H and $P(s_H)$ were argued to satisfy the scaling behavior, $s_H \simeq N^x$ and $P(S_H) \simeq N^{-y}$ with $x > 0$ and

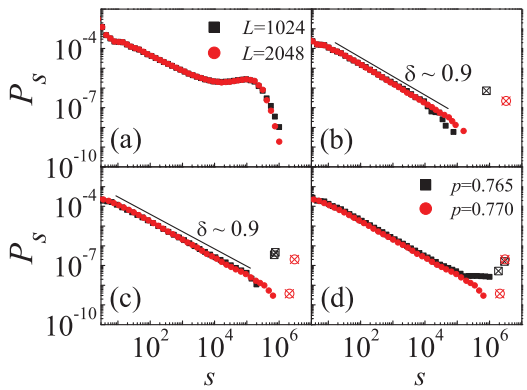

FIG. 1. (Color online) (a) Plot of P_s for $p = 0.75$ or $p < p_c$. The data for $L = 1024$ are denoted by black squares and those for $L = 2048$ are denoted by red circles. (b) Plot of P_s for $p = 0.8$ or $p > p_c$. The solid line represents the relation $P_s \sim s^{-\delta}$ with $\delta \simeq 0.90(2)$. The data points marked by "×" represent the P_s for macroscopically large clusters or the largest cluster. (c) The same plot for $p = 0.77$ or near the transition threshold. (d) Plot of P_s for $p = 0.765$ and $p = 0.770$ on the lattice with $L = 2048$.

$y > 0$ [15]. Therefore in APCG the hump has the negligible contribution and P_s satisfies the same scaling form as Eq. (2) in the limit $N \to \infty$. This P_s behavior in APCG is believed to be one of the signals for the continuous transition as in OP. In contrast s_H and $P(s_H)$ of 2DSAP do not depend on L or $s_H \simeq$ const and $P(s_H) \simeq$ const as $L(N)$ gets larger. We have numerically checked this behavior for $N = 2^{16}, 2^{18}, 2^{20}, 2^{22}$. This behavior for 2DSAP means that there should be many considerably large stable microscopic s clusters with $s \simeq s_H$ before transition, which indicates the unstable or sudden appearance of the macroscopic cluster by connecting these clusters when p increases.

Even when $p > p_c$, P_s for 2DSAP has a unique behavior as shown in Fig. 1(b). Except P_s for the macroscopically large cluster, P_s for microscopically finite clusters for $p > p_c$ still satisfies the same power law $P_s = As^{-\delta}$ with the same exponent δ as $P_s(p_c)$ or $\delta = 0.90(3)$, which we will explain with the data in Fig. 1(c). The difference between $P_s(p > p_c)$ and $P_s(p_c)$ is in A and the tail part for finite clusters. As p becomes larger than p_c, A decreases and the length of tail becomes shorter compared to P_s at p_c. The power-law behavior is very robust, because it maintains for nearly four decades as shown in Fig. 1(b) before appearing finite-size effects. Moreover, the power-law behavior for $p > p_c$ is nearly independent of L as for $p < p_c$ [see Fig. 1(b)]. This power-law behavior for the finite clusters has been confirmed even for large p up to $p = 0.9$. In contrast P_s of microscopic clusters for $p > p_c$ in OP and APCG exponentially decays. In 2DSAP the product rule makes the macroscopic cluster absorb relatively smaller clusters when p gets large in the regime $p > p_c$. Therefore the larger microscopic clusters cannot easily disappear. The sustainability of such metastable clusters seems to be the origin of the power law of P_s for $p > p_c$. As we shall see, the hysteresis of 2DSAP is consistent with the power law for $p > p_c$ because of such metastable states.

The phase transition for 2DSAP naturally occurs at p, which divides the two regimes of P_s described by Figs. 1(a) and 1(b). The transition threshold p_c for 2DSAP is estimated from the data sets as in Figs. 1(c) and 1(d) by identifying the p at which the hump disappears [22]. As shown in Figs. 1(c) and in 1(d) at $p = 0.770$, P_s for the macroscopically large cluster starts to be detached from the continuous distribution of P_s for microscopic clusters. This detachment behavior seems to be independent of $L(N)$ as shown in Fig. 1(c). As shown in Fig. 1(d), this detachment behavior barely occurs and the humplike tail still exists for $p = 0.765$. We have scrutinized P_s between $0.765 < p < 0.770$, but the sharp discrimination between the humplike behavior and the detachment cannot be made. Such complex behavior mixing the hump and P_s for the macroscopically large clusters for $p \simeq p_c$ seems to be a unique behavior of 2DSAP. Therefore the best estimation of p_c from the numerical data of P_s is $p_c = 0.768(3)$. At $p \simeq p_c$, P_s satisfies the power law $P_s = As^{-\delta}$ with $\delta = 0.90(2)$ very well. Again this power law $P_s = As^{-0.9}$ is very robust and holds for more than four decades. The result $\delta = 0.9$ also provides a very important clue to understand the transition nature of 2DSAP. Since P_s is a probability, P_s should satisfy the normalization condition $\sum_s P_s = p$. However, the summation $\sum_s^\infty P_s$ diverges if $\delta < 1$. Therefore there should

be a cutoff s_c in the upper limit as $\sum_s^{s_c} P(s) = p$. In the limit $N \to \infty$, $s_c/N \to 0$. Thus there should be a discontinuous jump to produce a macroscopic cluster in the limit $N \to \infty$. The physical origin of the discontinuous transition should come from the merge of s clusters with $s \simeq s_h$ to form the macroscopic cluster when p gets larger to be $p = p_c$.

One of the most generally accepted and simplest methods to verify whether the observed transition is discontinuous or not is the measurement of the hysteresis [23]. The hysteresis measurement for the explosive percolation has also been emphasized in Refs. [11,20]. If the transition is discontinuous, then the route of changes in the order parameter P_{LC} during the process of filling sites from the vacant lattice or increasing p would be different from that for the process of deleting sites from the fully occupied lattice or decreasing p. The order parameter P_{LC} is defined by the probability that a site belongs to the largest cluster [1,8]:

$$P_{LC} = \frac{N_{LC}}{N}. \tag{3}$$

Here, N_{LC} is the number of sites in the largest cluster. In Fig. 2(a), we compare the measured P_{LC}'s along the process of increasing p (solid lines) and along the process of decreasing p from $N = 2^{12}$ to $N = 2^{20}$. For the decreasing process, we slightly modify rule (III) to easily break the larger clusters into smaller ones [11], since rules (II) and (III) suppress the formation of a large cluster; i.e., if $\pi_\alpha \geqslant \pi_\beta$ then we delete the site α. With this modified rule we find that there exists a hysteresis for various L as shown in Fig. 2(a).

Now the remaining question is whether the hysteresis robustly remains in the $L \to \infty$ limit. For the systematic analysis, we show the dependence of area $A(L)$ enclosed by $P_{LC}(L)$ for the increasing and decreasing processes on L. If the system undergoes a continuous transition, then $A(L)$

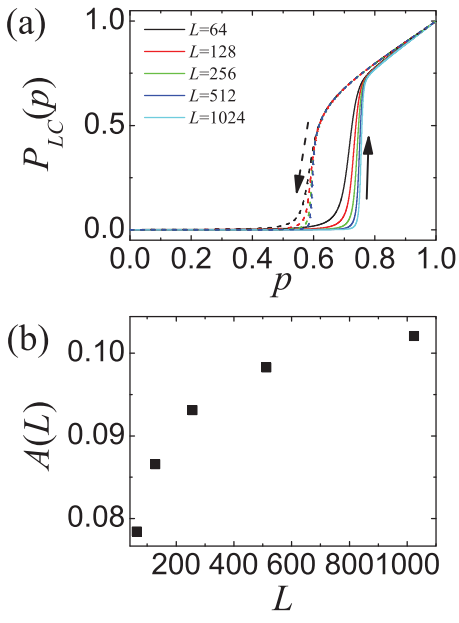

FIG. 2. (Color online) (a) Plot of $P_{LC}(p)$ for the process of increasing p (solid line) and that for the process of decreasing p (dashed line). L varies from 64 (left most line) to 1024 (right most line). (b) Plot of the area $A(L)$ enclosed by P_{LC}.

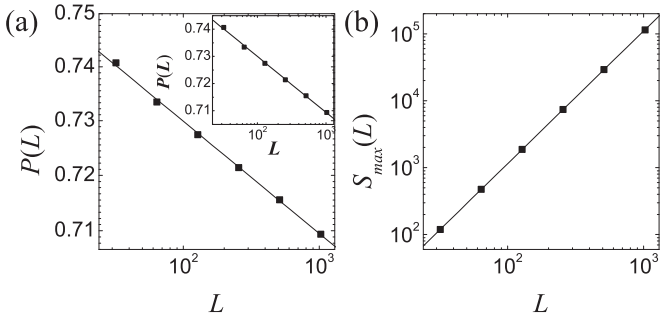

FIG. 3. (a) Plot of $P_{LC}(L)$ measured at p_c against L. The solid line represents the relation $P(L) \sim L^{-B}$ with $B = 0.012$. Inset: Plot of $P_{LC}(L)$ in semilog scale. The solid line represents the relation $P(L) \sim -\log L$. (b) Plot of $S_{\max}(L)$ against L. The solid line denotes $S_{\max} \sim L^C$ with $C = 1.98$.

should vanish in the limit $L \to \infty$. However, our data clearly show that $A(L)$ increases as L increases or, at least, seems to saturate to a nonzero value unlike the sum rule [20] in which $A(L) \to 0$ as $L \to \infty$. This shows that 2DSAP undergoes a discontinuous transition. And in a 2D lattice, the product rule makes a completely different transition nature from that of the sum rule [20]. This hysteretic property of 2DSAP should be from the sustainability of the metastably larger clusters, which is consistent with the analysis of P_s in Fig. 1.

Since we do not know the physically corresponding formula to Hamiltonian or free energy for 2DSAP and there exists the nontrivial hysteretic property, it might be physically nonsensical to discuss the finite-size scaling. However for the purpose of comparison to other works on explosive percolation [8–10], we now present the finite-size analysis around $p_c = 0.768(3)$, which is the percolation transition probability for the p-increasing process. From the data in Fig. 2(a), $P_{LC}(L)$ at the p_c is estimated as shown in Fig. 3(a). $P_{LC}(L)$ seems to satisfy the relation $P(L) \sim L^{-B}$ with $B = 0.011(2)$, where conventionally B corresponds to β/ν. This value of B is very close to zero. Thus in the inset of Fig. 3(a) we also fit the data to the relation $P(L) \sim -\log L$, which corresponds to the case $B \to 0$. Since we cannot exclude the possibility $B = 0$ or $\beta = 0$, the possibility for $P_{LC}(L \to \infty)$ at p_c to have discontinuous jump cannot be excluded. We also measure the mean cluster size, defined by

$$S(p,L) = \frac{\sum_s' s P_s}{\sum_s' P_s}. \tag{4}$$

$\sum_s'$ represents the summation over all s except the largest one. $S(p,L)$'s maximal value $S_{\max}(L)$ is displayed in Fig. 3(b). Again we fit the data to the conventional scaling relation $S_{\max}(L) \sim L^{-C}$, and we obtain $C = 1.98(1)$, where C corresponds to conventional γ/ν.

In summary, we study the site percolation under AP with a product rule in a 2D lattice. From the measurement of $P_s(p)$, we find that $P_s(p)$ has a very stable hump when $p < p_c$. This indicates that below p_c a large number of stable s clusters with $s \simeq s_H$ exist but their sizes are still microscopic. The similar behaviors near p_c were reported in other discontinuous percolation models [24]. As p approaches p_c, $P_s(p)$ has a very robust power-law regime followed by

the hump. Since the obtained value of the exponent δ for the power-law regime in the vicinity of p_c is less than unity, there should be a cutoff s_c in the possible cluster size for $p \simeq p_c$ unlike OP [1]. Thus to generate a macroscopic cluster there should be a discontinuous jump in the limit $L \to \infty$ and the transition becomes discontinuous. The non-vanishing hysteresis in P_{LC} also verifies that the transition is discontinuous. This result clearly shows that the percolation transition caused by the product rule in a 2D square lattice is discontinuous.

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