Physica A 390 (2011) 4034–4037

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Self-organized scale-free networks generated via Merging-and-Creation dynamics with preferential attachment

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ARTICLE INFO

Article history:
Received 30 August 2010
Received in revised form 16 June 2011
Available online 26 June 2011

Keywords:
Networks
Merging-and-Creation dynamics
Scale-free networks

ABSTRACT

We study a self-organized scale-free network model generated using the Merging-and-Creation dynamics with preferential attachment. We show analytically that the introduction of preferential attachment has minimal impact on the steady-state degree distribution. However, we find also that the preferential attachment gives rise to a hierarchical modular structure and degree disassortativity, commonly found in technological networks.

A wide range of complex networks exhibit power-law or scale-free (SF) distributions of connectivity between constituent nodes [1–3]. Various models have been proposed to explain the phenomenon, *preferential attachment* being one of the most studied [2–4]. Other approaches include an explicit construction of graph Hamiltonians [5–8] and the self-organized criticality (SOC) network model [9]. The SOC network model, based on the Merging-and-Creation (MC) dynamics, was originally proposed as the underlying mechanism of the power-law distribution of magnetic flux loops and the flare energy of the Sun [9].

The MC process is perhaps the simplest model of offspring generation through random mating in biological systems. When the model is applied to networks [10–12], the Simple MC (SMC) model consists of merging two randomly chosen nodes, and creating a new node that connects to a pre-existing node at each time step. It is equivalent to the mass aggregation model of Takayasu [13], and analytical and numerical studies have shown that the resulting degree distribution follows a power law $P(k) \sim k^{-\gamma}$ with $\gamma = 3/2$, identical to the mass distribution from the mean-field solution of the Takayasu model [13].

The degree exponent $\gamma = 3/2$ we find from the MC model is very interesting because, while several real-world networks such as the e-mail [14] and the software networks [15,11] exhibit exponents smaller than 2, such a case has not been studied as much as the more common cases of $\gamma > 2$. Recently, Seyed-allaei et al. studied SMC model to investigate the properties of the link weights in networks. Their model captures the evolutionary features of each programming unit in software networks [11]. Even more recently, we studied the detailed structure of the SMC network model, including the hierarchical modular structure and degree–degree correlations, and compared them with those of a Hamiltonian-based network model [8]. There we found that the SMC process was not sufficient to reproduce said features of real-world networks [15], and discussed the effect of young nodes on the differences. In this report, as an extension of the previous work, we study the effect of interactions between the young and the old node in a network generated from a modified MC process in which preferential attachment is used in the creation process, encouraging newly created nodes to connect to high-degree (i.e. old) nodes more strongly than the SMC model. We show analytically that a power-law degree distribution

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doi:10.1016/j.physa.2011.06.038

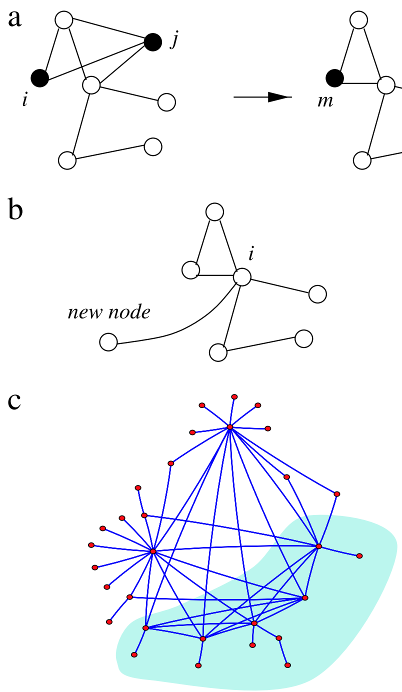

Fig. 1. (Color online) Schematics of the Merging–Creation dynamics with preferential attachment. (a) *Merging*: two randomly selected nodes i and j are merged into a single node m. (b) *Creation*: a new node is linked to one of the existing nodes with probability proportional to the existing node's degree (preferential attachment). Therefore, the most probable choice here is node i with degree $k = 4$. (c) Sample network configuration with $N = 40$ nodes. The nodes in the shaded area are relatively densely interconnected.

results with exponent γ equal to that from the SMC model despite the addition of preferential attachment. We also show, nevertheless, that the effect of preferential attachment is manifest in other topological properties of the network [8].

Starting from a regular ring of N nodes, each connected to its $\bar{k}_0$ nearest neighbors, the MC process with preferential attachment consists of the following procedures at each step: (1) *Merging*: two randomly chosen nodes i and j merged into a single node m of degree $k_m = k_i + k_j - N_{common}$ where N_{common} is the number of common neighbors of nodes i and j before the merger plus the number of connection between themselves (0 or 1) (Fig. 1(a)). (2) *Creation*: a new node is created and linked to a pre-existing node i (not to be confused with node i in the merging step) with probability (Fig. 1(b))

$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}. \tag{1}$$

In this model the number of nodes stays constant at N, and in the steady state the number of total edges is nearly constant. All the network properties in this report are measured in the steady state. As discussed in Ref. [8], even in the SMC model, very old nodes tend to be connected to young nodes (usually ones with degree 1). Thus we expect that with the preferential attachment such effect is even stronger (Fig. 1(c)).

We start by deriving the degree distribution analytically. The rate equation for the average number $N(k, t)$ of nodes of degree k at time t is

$$N(k, t+1) = N(k, t) + \delta_{k,1} + \frac{k-1}{N\bar{k}} N(k-1, t) - \frac{k}{N\bar{k}} N(k, t) + \left(\frac{1}{N}\right)^2 \sum_{k'+k''=k} N(k', t) N(k'', t) - \frac{2}{N} N(k, t), \tag{2}$$

where $\bar{k}$ is the average degree of the network. The first four terms on the right-hand side of Eq. (2) represent the creation of a new node with preferential attachment, while the last two represent the merging of two randomly chosen nodes. Eq. (2) is similar to "network A" in Ref. [12] except for the coefficients that represent preferential attachment. We assume that the network is sufficiently large and sparse that we can safely ignore the effects of common neighbors and multiple connections during the merging step.

From Eq. (2), the evolution equation for the degree distribution, $P(k, t) = N(k, t)/N$ becomes

$$\frac{\partial P(k, t)}{\partial t} = \delta_{k,1} + \frac{k-1}{\bar{k}} P(k-1, t) - \left(\frac{k}{\bar{k}} + 2\right) P(k, t) + \sum_{k'+k''=k} P(k', t) P(k'', t). \tag{3}$$

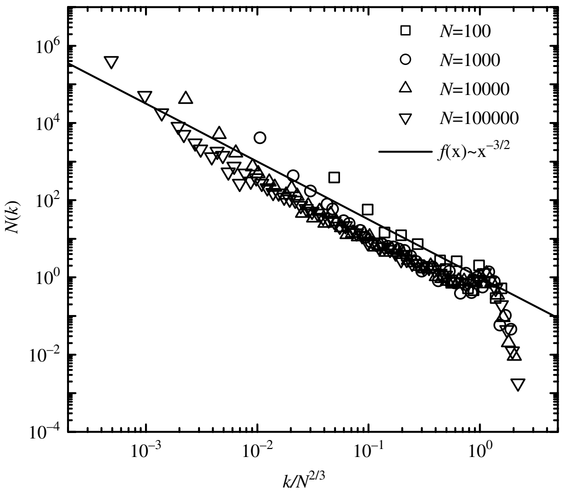

Fig. 2. Plot of rescaled $N(k)$ against $k/N^{2/3}$ for $N = 10^2$, 10^3, 10^4, and 10^5. The solid line represents $N(k) \sim k^{-\gamma}$ with $\gamma = 3/2$.

Eq. (3) in the steady state where $\partial_t P(k, t) = 0$ can be solved using the Z-transformation (generating function) method [16,12]. The Z-transformation of the degree distribution is defined as (we use $P(k)$ in place of $P(k, t)$)

$$G_0(z) \equiv \sum_{k=0} z^k P(k), \tag{4}$$

with which we can rewrite Eq. (3) as

$$G_0^2(z) - 2G_0(z) + z + \frac{z(z-1)}{\bar{k}} = 0, \tag{5}$$

which leads to

$$G_0(z) \simeq 1 \pm (1-z)^{1/2}\left(1 + zG_0'(z)/\bar{k}\right)^{1/2}.$$

Using to the relation

$$\lim_{k\to\infty} P(k) \sim k^{-\gamma} \to \lim_{z\to 1} G_0(z) \simeq c_1 + c_2(1-z)^{\gamma-1} + \text{analytic terms}, \tag{6}$$

we obtain the degree exponent $\gamma = 3/2$ [12]. This is identical to the result of SMC, meaning that the degree distribution is not significantly affected by preferential attachment at all. We confirm this result via numerical simulation: Fig. 2 shows $N(k)$ for various network sizes ($N = 10^2$–10^5). Note that $\gamma < 2$ implies that the average degree $\bar{k}$ increases with the system size $\bar{k} \sim N^{\xi}$ and the structural cutoff scales as $k_c \sim N^{(1+\xi)/2} \sim N^{\sigma}$. An explicit calculation of $\bar{k}$ predicts that ξ satisfies the relation $\xi = (2-\gamma)/\gamma$ when $\gamma < 2$ [11]. Therefore, $N(k)$ scales as

$$N(k) \equiv N \times P(k) \sim Nk^{-\gamma} f\left(\frac{k}{N^{\sigma}}\right), \tag{7}$$

where $f(x)$ is a scaling function that is $\sim$*const* for $x \ll 1$ and decays faster than any power of x when $x \gg 1$ [11]. Using exponent $\sigma = 2/3$, we see that $N(k)$ for various N collapse well. The solid line represents the relation $N(k) \sim k^{3/2}$. This shows that the analytical solution (Eqs. (6) and (7)) is in good agreement with the simulations.

The clustering coefficient C is an important indicator of the existence of the hierarchical structure. It is well known that in a network lacking any hierarchical structure C does not depend on k, whereas in a network with well-defined hierarchical modules the local clustering coefficient $C(k)$ is a decreasing function of k, i.e. $C(k) \sim k^{-\beta}$ [17]. In the SMC model (with no preferential attachment), $C(k)$ is known to be an increasing function of k [8]. However, as shown in Fig. 3(a), $C(k)$ under the MC model with preferential attachment is only a slightly increasing (almost flat) function for k ($<10^3$ for $N = 10^5$). Near k_{max}, $C(k)$ shows a clearly decreasing tail approximated as $C(k) \sim k^{-\beta}$ with $\beta = 1.5$. This is similar to that of the equilibrium model studied in Ref. [8]. This implies that the preferential attachments of new nodes result in a more hierarchical modular structure than the one found in the SMC network model.

The degree–degree correlation is another important measure with which one can characterize the network structure. According to the sign of the correlation, networks can be classified into two groups: a networks with positive (negative) correlation is said to show assortative (disassortative) mixing [18]. In general, social networks show assortative mixing while technological networks (e.g. WWW and the Internet) show disassortative mixing. Several topological and dynamical properties are known to be affected by the degree correlations [19,20]. We investigate the property of degree–degree

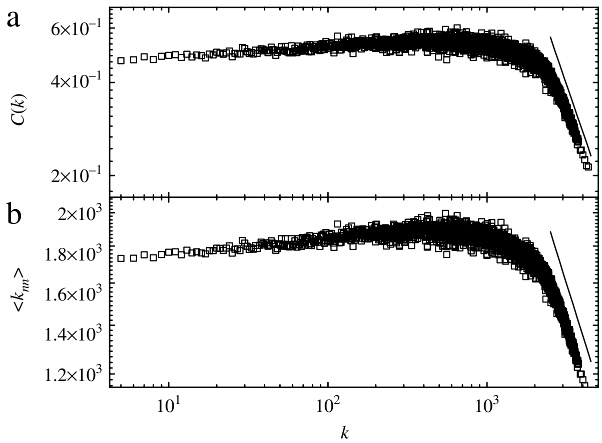

Fig. 3. Plot of (a) $C(k)$ and (b) $\langle k_{nn}(k)\rangle$ of the MC with preferential attachment model for $N = 10^5$. The solid lines represent (a) $C(k) \sim k^{-1.5}$ and (b) $\langle k_{nn}(k)\rangle \sim k^{-0.7}$, respectively.

correlation by measuring the $\langle k_{nn}(k)\rangle$, the average degree of the neighbors of nodes that have degree k. Unlike the SMC model [8] where the degree correlation shows a strong assortative mixing for small k, $\langle k_{nn}(k)\rangle$ of the MC model with preferential attachment slightly increases (but almost flat) for small k (Fig. 3(b)). However, near k_{max}, $\langle k_{nn}(k)\rangle$ also exhibits an explicitly decreasing tail. This implies that the tendency of young nodes to connect to old nodes are stronger, clearly an effect of preferential attachment combined with aging. As discussed in Ref. [8], even in the SMC model old nodes can become hubs. The preferential attachment drives such tendency further by enhancing connections between high- and low-degree nodes (for example see Fig. 1(c)), driving the degree correlation to be more negative.

In this study, we explored the characteristics of the MC model with preferential attachment. Through analytic derivation and numerical simulation we showed that the degree exponent γ is not affected by the preferential attachment. However, we also found that the preferential attachment causes a hierarchical modular structure to emerge between high- and low-degree nodes. We also find that the degree correlation becomes negative (or disassortative) for large k. From these results, we expect that many technological networks such as the software network [15] could be better modeled via the MC process with preferential attachment.

Acknowledgments

This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (KRF-20110005499, KRF2009-0073939, MOEHRD), and by the Kyung Hee University Grant KHU-20110088.

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