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Effects of the underlying topology on perturbation spreading in the Axelrod model for cultural dissemination

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ABSTRACT

We study the effects of the underlying topologies on a single feature perturbation imposed to the Axelrod model of consensus formation. From the numerical simulations we show that there are successive updates which are similar to avalanches in many self-organized criticality systems when a perturbation is imposed. We find that the distribution of avalanche size satisfies the finite-size scaling (FSS) ansatz on two-dimensional lattices and random networks. However, on scale-free networks with the degree exponent $\gamma \leq 3$ we show that the avalanche size distribution does not satisfy the FSS ansatz. The results indicate that the disordered configurations on two-dimensional lattices or on random networks are still stable against the perturbation in the limit N (network size) $\rightarrow \infty$. However, on scale-free networks with $\gamma \leq 3$ the perturbation always drives the disordered phase into an ordered phase. The possible relationship between the properties of phase transition of the Axelrod model and the avalanche distribution is also discussed.

1. Introduction

Recently, there have been great attempts to understand how self-organization, adaptation, and cooperation arise various complex patterns in social systems [1–3]. The basic concepts and tools developed in statistical mechanics and nonlinear dynamics have been shown to be useful to find the general mechanism behind social systems [2]. In particular, the spreading of culture and the formation of cultural domains have attracted many physicists due to their complex behaviors and the relevance to the order–disorder transition in statistical mechanics [4–8]. Here, the culture is defined by the set of individual attributes or features, such as belief, opinion, language and social norm. The cultural domain is defined by the group of people who share the same cultural traits. In order to study the observed behaviors and the related phenomena in the spreading and formation of cultural domains, numerical simulations of simple dynamical models have been used [2].

In this spirit, Axelrod introduced a simple model to investigate how the dissemination and formation of cultural domains arise [4,5] based on two assumptions: (i) individuals more likely interact with others if they share more of their cultural attributes; (ii) whenever the interaction occurs the number of sharing features between the interacting individuals increases. Thus the interaction always decreases the cultural differences between interacting individuals. These assumptions qualitatively resemble the traditional models of statistical physics, in which the local interaction tends to decrease the total energy. However, Axelrod showed that the local convergence by interaction causes social cleavages which lead to a global polarization when the cultural traits are diverse enough [4]. More precisely, in the Axelrod model each individual is placed on each site of a lattice. The cultural attributes of each individual are modeled by a set of F features. Each feature takes one of

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the Q possible integer values which represents a cultural trait. Recent studies have shown that there exists a certain value Q_c at which the system undergoes a order–disorder transition. When $Q < Q_c$ the system is in an ordered phase (monocultural phase) and when $Q > Q_c$ it is in a disordered phase (multicultural phase). [6,8].

The study on the stability of the disordered phase has theoretical and practical importance as addressed in Ref. [9]. Recent study revealed that the competition between the single feature perturbation rate and the relaxation time plays an important role to determine the steady state configuration of the Axelrod model against the perturbation [9]. When the perturbation rate is smaller than the inverse of the characteristic time, a disordered phase in a finite two-dimensional system evolves into an ordered phase. On the other hand, when the perturbation rate is larger than the inverse of the characteristic time, the disordered phases are stable against the perturbation. This result implies that when there is enough time to spread the applied perturbation, the repeated action of these perturbations drives the disordered phase into an ordered phase if the system is finite. Thus, the disordered phases are unstable in finite systems [9,10].

Like many other phase transitions [11–13], the property of phase transition of the Axelrod model is also crucially affected by underlying topologies. In particular, on lattices and random networks (or small-world networks) [14,15] the transition occurs at a finite Q_c when the network size, $N \to \infty$. However, on the scale-free (SF) networks with a power-law degree distribution, $P(k) \sim k^{-\gamma}$, Q_c diverges in the limit $N \to \infty$ and the model is always in the ordered state when $\gamma = 3$ [7]. Moreover, many social and economical interactions are known to form complex networks (for example, see the Ref. [16]). Therefore, investigating the effect of underlying topology on the formation and spreading of cultural domains is very important to understand the dynamical properties of various pattern formations in the real world.

So far, there is no systematic study on the processes of how the repeated action of single feature perturbations spreads over the entire system and whether finally the system reaches an ordered state or not. In this paper we investigate the effect of the various underlying topologies on the spreading of the repeated action of single feature perturbation and show that the spreading is affected by the underlying topology. For this purpose we use complex networks as well as two-dimensional (2D) square lattices. In the vicinity of the observed Q_c, the disordered configuration of the Axelrod model has various sizes of different cultural domains [6]. In this study, we show that the successive applications of single feature perturbations cause various sizes of avalanches which merge the different domains into a single ordered domain in finite systems. Based on the finite-size scaling analysis of avalanche size distribution, we find that the Axelrod model on 2D square lattices and random networks requires avalanches of infinite size to reach an ordered phase when $N \to \infty$. However, on SF networks with $\gamma \leq 3$, we show that the disordered configuration can evolve into an ordered configuration with only finite avalanches. As we shall show, these different behaviors on various topologies are closely related to the transition property of the Axelrod model.

The paper is organized as follows. In Section 2 we present the definition of the original Axelrod model and single feature perturbation model. The used underlying networks are also provided in Section 2. The numerical results for the spreading of single feature perturbations are presented in Section 3, and summaries and discussions are in Section 4.

2. Model definitions

2.1. Original Axelrod model

The Axelrod model [1,4] on complex networks is defined as follows [7]: Each individual is located at each node in the network of size N. The state of each agent i is characterized by an F component vector (cultural features), $\{\sigma_{if}\}$, where $f = 1, \ldots, F$. Initially, an integer in the interval $[1, Q]$ is randomly assigned to each σ_{if}. At each time step, the state of each agent is updated by the following dynamic rule: (1) Select a directly connected pair of nodes (i, j) at random. (2) Calculate the overlap (or the number of the shared features) between agents i and j, $\ell(i, j) = \sum_{f=1}^{F} \delta_{\sigma_{if},\sigma_{jf}}$. Here, $\delta_{n,m}$ represents the Kronecker's delta. (3) If $0 < \ell(i, j) < F$, then the sites i and j are said to be active. These active sites interact to each other with probability ℓ/F. In the case of interaction, choose the f'th component at random among those having different values between i and j (i.e., $\sigma_{if'} \neq \sigma_{jf'}$) and update $\sigma_{if'} = \sigma_{jf'}$. (4) If $\ell = 0$ or $\ell = F$ then the bond (i, j) becomes inactive and nothing happens. In the following simulations we fix the value of F to be 10.

In any finite system, the dynamics of the Axelrod model leads to an absorbing state, characterized by the absence of active bonds. In the Axelrod model, two different absorbing states are possible. When Q is small, each agent can interact to its nearest neighbors with high probability. As a result, the system evolves into an ordered or monocultural state ($\sigma_{if} = \sigma_{jf}, \forall (i, j), \forall f$) for small Q. On the other hand, when Q is large, the interaction probability (or ℓ) significantly decreases, and the system settles into a disordered or multicultural state.

2.2. Single feature perturbation and relaxation (SFPR)

In order to investigate the stability of a disordered phase in the Axelrod model, a single feature perturbation model was introduced [9]: (i) repeat the dynamic processes (1)–(4) in Section 2.1 until the system reaches an absorbing state. (ii) A node i is randomly selected. Then a randomly selected component of node i, σ_{if}, is changed to an arbitrary value s ($\in [1, Q]$) (single feature perturbation). (iii) Repeat the procedure (i)–(ii) (relaxation). The perturbation–relaxation procedures cause successive updates.

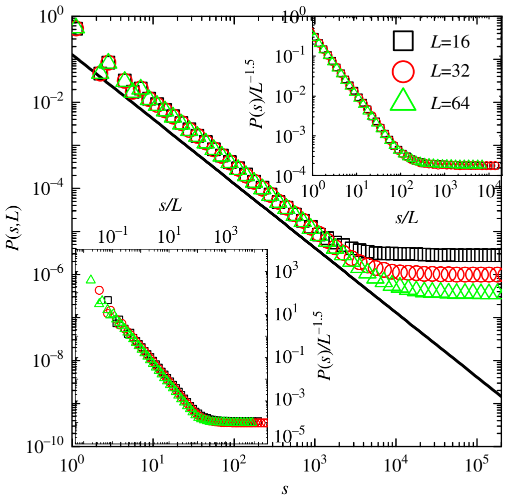

Fig. 1. (Color online) Plot of $P(s, L)$ for various L on two-dimensional lattices when $F = 10$ and $Q = 60$. The solid line represents the relations $P(s) \sim s^{-1.5}$ when $Q = 60$. Upper inset: scaling plot of $P(s, L)$ for $Q = 60$ with $\nu = 1$ and $\tau = 1.5$. Lower inset: scaling plot of $P(s, L)$ for $Q = 200$ with $\nu = 1$ and $\tau = 1.5$.

2.3. Underlying networks

In the following simulations, we consider two types of different network topologies, random networks and SF networks, as well as a 2D square lattice. For the construction of the random network, we use the Erdős–Rényi (ER) network model [14]. The degree distribution of ER networks is known to satisfy the Poisson distribution which indicates that the degree distribution is homogeneous. In contrast to ER networks, SF networks show high heterogeneity in the degree distribution. In many systems, such as the Ising model, the critical behaviors are crucially affected by the topological heterogeneity [17]. In order to generate such SF networks with tunable γ, we use the static model suggested by Goh et al. [18]. In this model, a weight $w_i = i^{-\alpha}$ is assigned to each node i ($i = 1, 2, \ldots, N$), where $0 \le \alpha < 1$. By adding a link between unconnected nodes i and j with probability $w_i w_j / (\sum_{n=1}^{N} w_n)^2$, one can obtain a network whose degree distribution satisfies a power-law $P(k) \sim k^{-\gamma}$. In the static SF network model, γ is related to α as $\gamma = (1 + \alpha)/\alpha$. Thus, by adjusting α we obtain a network with any γ (>2).

3. Avalanche distribution of SFPRs

3.1. On a two-dimensional square lattice and on random networks

On one- and two-dimensional lattices, only the completely ordered configurations are known to be stable and other absorbing configurations are unstable [9]. Klemm et al. [9] numerically showed that under the repeated action of SFPRs the disordered phase evolves into an ordered phase in the finite-size systems. Their measurement also showed that there exist many perturbation–relaxation procedures between the changes in order parameter. This is reminiscent of the punctuated equilibria in self-organized criticality (SOC) [19].

From the numerical simulations we find that the SFPR causes successive updates of neighboring sites until the system reaches the other absorbing configuration. Such a successive update is usually called an avalanche. The avalanches in the perturbed Axelrod model play a very crucial role to drive the system into an ordered state. For the systematic analysis we measure the distributions of avalanche sizes from the avalanches which occur during the repeated applications of SFPRs to an initial disordered phase to reach the final absorbing phase. Here, the avalanche size, s, is defined by the total number of updates in a given SFPR. In the perturbed Axelrod model on 2D lattices, we find that the distribution of s satisfies the finite-size scaling (FSS) ansatz:

$$P(s) = s^{-\tau} \mathcal{F}\left(\frac{s}{s_c}\right). \tag{1}$$

Here $\mathcal{F}(x)$ represents a universal scaling function. When the linear dimension, L, of the underlying lattice goes to infinity, the cutoff s_c diverges as $s_c \sim L^{\nu}$. Eq. (1) can be rewritten as

$$P(s, L) = L^{-\nu\tau} f\left(\frac{s}{L^{\nu}}\right), \tag{2}$$

where $f(x)$ is another universal function. The universal function $f(x)$ scales as $f(x) \sim x^{-\tau}$ for $x \ll 1$ and saturates to a constant value when $x \gg 1$ [19]. Fig. 1 shows the simulation results of the perturbed Axelrod model on 2D lattices when

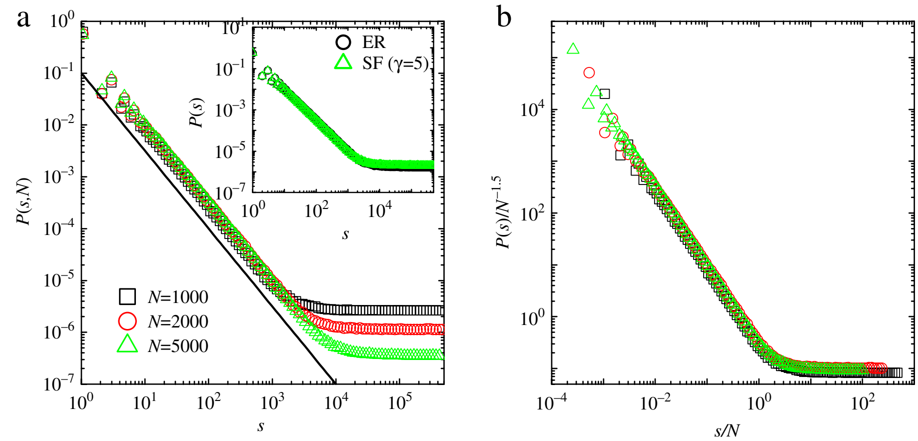

Fig. 2. (Color online) (a) Plot of $P(s, N)$ on ER networks for various N. The solid line denotes the relation $P(s) \sim s^{-1.5}$. Inset: Plot of $P(s)$ measured on the ER network and SF network with $\gamma = 5$ when $N = 1000$. (b) Scaling plot of $P(s, N)$ with $\bar{\nu} = 1$ and $\tau = 1.5$.

$F = 10$ and $Q = 60$. With these values of F and Q, the unperturbed Axelrod model on the 2D lattice is known to be in the disordered phase [7]. From the data in Fig. 1 we obtain $\tau = 1.50 \pm 0.01$ and $\nu = 0.99 \pm 0.01$. The upper inset of Fig. 1 shows that the scaling plot of $P(s, L)$ using the obtained τ and ν satisfies the relation (2) very well. The lower inset shows the same plot for $F = 10$ and $Q = 200$, which shows that for any value of Q $(>Q_c)P(s, L)$ satisfies Eq. (2).

In Fig. 2 we show the $P(s, N)$ measured on ER networks with $\langle k \rangle = 4$, $F = 10$ and $Q = 200$. Here we use the size of the network, N, for the finite-size scaling analysis instead of the linear dimension L of the 2D lattice. The used values of F and Q insure that the system is in the disordered configuration when the dynamic rules (1)–(4) are accomplished (for example, see Fig. 5(a)). As shown in Fig. 2, we find a similar scaling behavior of $P(s, N)$ in the perturbed Axelrod model on the ER networks with Eq. (2) as Ref. [20]

$$P(s, N) = N^{-\tau\bar{\nu}} f(s/N^{\bar{\nu}}). \tag{3}$$

From the data in Fig. 2(a) we obtain $\tau = 1.51 \pm 0.02$ and $\bar{\nu} = 0.99 \pm 0.07$. In Fig. 2(b) we display the scaling plot of $P(s, N)$ using the obtained τ and $\bar{\nu}$, which verifies the FSS ansatz (3).

Note that $P(s \to \infty, L)$ and $P(s \to \infty, N)$ decrease as L and N increase, respectively (see Figs. 1 and 2). This implies that in order to reach the ordered phase the dynamic procedure requires avalanches whose sizes are larger than L or N. However, the probability for such a large avalanche to occur approaches to 0 as L or N increases. Therefore, in the limit $L \to \infty$ or $N \to \infty$, the system cannot reach the ordered phase and the disordered phase becomes stable against the repeated application of SFPRs on 2D lattices or ER networks. We find the same results on SF networks with $\gamma > 3$ (see the inset of Fig. 2(a)) like the other critical phenomena [17] and dynamical processes [21,22].

3.2. On SF networks with $\gamma \leq 3$

On SF networks with $\gamma \leq 3$, we find completely different scaling behaviors of $P(s, N)$ from those on 2D lattices and ER networks. As shown in Fig. 3, we find that $P(s, N)$ decays as

$$P(s) \sim s^{-\tau}, \tag{4}$$

with $\tau_s = 1.49 \pm 0.01$ when $s < 10^4$. Then, $P(s)$ saturates to the same constant value for $s > 10^4$, regardless of N. To satisfy the normalization condition, $P(s)$ for $s > 10^4$ should have a cutoff $s_{\rm cut}$ which satisfies $P(s > s_{\rm cut}) = 0$ in the limit $N \to \infty$ and $\bar{\nu}$ becomes zero. This indicates that in order to reach an ordered configuration on SF networks, the dynamics of the perturbed Axelrod model requires only a finite size of avalanches even in the limit $N \to \infty$. Thus, the repeated SFPRs always drive the system into an ordered phase on SF networks with $\gamma \leq 3$. As we will show in Section 3.3, when $\gamma \leq 3$ the Axelrod model is always in the ordered phase in the limit $N \to \infty$. Therefore, the disordered phase observed in the finite N is unstable against the perturbation, and the system easily evolves into an ordered phase even when s is finite. Since the static SF networks are weakly disassortative when $\gamma \leq 3$ [23], we also measure the $P(s, N)$ on the assortative networks (see the inset of Fig. 3). The assortativity of networks is adjusted by the method in Ref. [24]. As shown in the data we find no difference in $P(s, N)$ between assortative and disassortative networks.

In order to verify the convergence of $s_{\max}$ on SF networks with $\gamma \leq 3$, we measure the average value of maximum avalanche size $\langle s_{\max} \rangle$. As shown in Fig. 4, $\langle s_{\max} \rangle$ abruptly increases when $N < 2000$ on SF networks with $\gamma \leq 3$. Then it saturates to some constant value ($\sim 10^6$) for $N \geq 2000$. This shows a clear agreement with the expectation based on the scaling behavior of $P(s, N)$ in Fig. 3 and Eq. (4). For a comparison, we also display $\langle s_{\max} \rangle$ on 2D lattices in the inset of Fig. 4 which clearly shows that $s_{\max}$ increases as L increases.

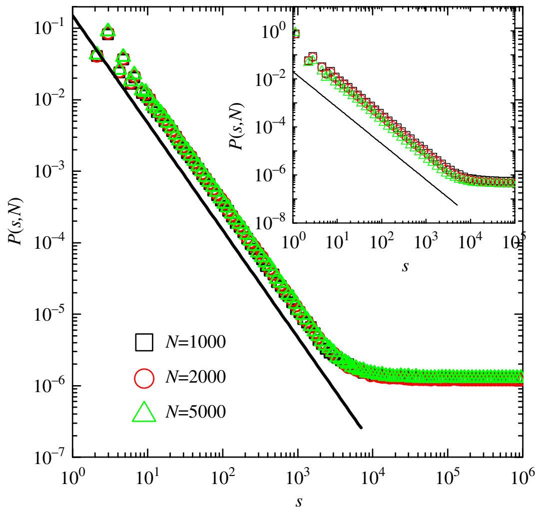

Fig. 3. (Color online) Plot of $P(s, N)$ on SF networks with $\gamma = 2.7$ when $F = 10$ and $Q = 200$. The solid line represents the relation $P(s) \sim s^{-1.5}$. Inset: Plot of $P(s, N)$ on assortative SF networks. The solid line corresponds to $\tau = 1.5$.

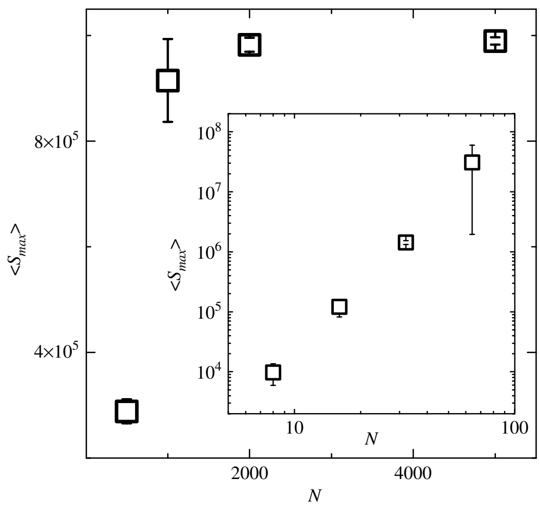

Fig. 4. Plot of $\langle s_{\max} \rangle$ for SF networks with $\gamma = 2.5$. Inset: Plot of $\langle s_{\max} \rangle$ for 2*D* lattices.

3.3. Origin of the differences in $P(s)$

In order to understand the observed differences in $P(s, N)$ on SF networks with $\gamma \leq 3$ and on 2*D* lattices or ER networks, we measure Q_c at which the phase transition of the original Axelrod model occurs. As shown in Fig. 5(a) and (b), $Q_c(N)$ on each network shows completely different behaviors. On ER networks (or SF networks with $\gamma > 3$), Q_c converges to a finite value in the limit $N \to \infty$. But on SF networks with $\gamma \leq 3$ we find that Q_c diverges as

$$Q_c \sim N^{\alpha}, \qquad (5)$$

where the exponent α increases as γ decreases. The result indicates that the system is always in the ordered phase when $\gamma \leq 3$ in the limit $N \to \infty$. Similar behavior was observed in the Ising model on SF networks [17]. For the Ising model, the critical temperature diverges in the limit $N \to \infty$ for $\gamma \leq 3$. On the other hand, the transition occurs at finite temperature when $\gamma > 3$. This behavior can be qualitatively understood from the topological properties of the underlying network. The effect of hubs becomes crucial when $\gamma \leq 3$ like the epidemic spreading [25]. Through the interactions, the nodes connected to the hubs follow the cultural traits of the hub to minimize the cultural differences. Thus, the largest hub becomes a cultural leader. Moreover, the ultrasmall-world property [26] observed when $2 < \gamma \leq 3$ can facilitate the spreading of cultural domains over the whole network. As a result, the Axelrod model is always in an ordered (monocultural) phase when $\gamma \leq 3$. On the other hand, when $\gamma > 3$ the degree of the hub in the SF network becomes relatively small compared to those for

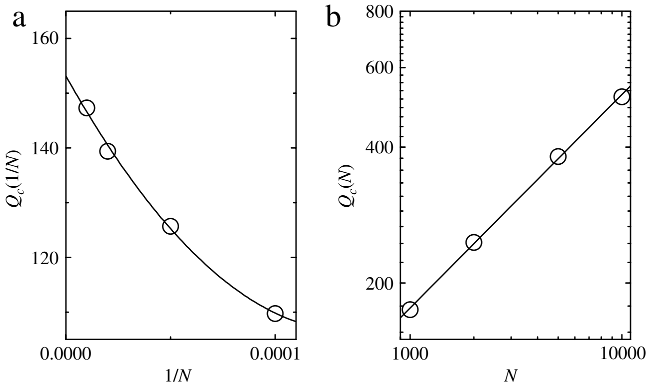

Fig. 5. (a) Plot of $Q_c(1/N)$ on ER networks. From the extrapolation of the data (solid line) we estimate a value of $Q_c(1/N \to 0) \approx 153$ for ER networks. Similar behavior of $Q_c(1/N)$ is observed for SF networks with $\gamma > 3$. (b) Plot of $Q_c(N)$ on SF networks with $\gamma = 2.7$. The solid line represents the relation $Q_c(N) \sim N^{\alpha}$ with $\alpha = 0.47$.

$\gamma \leq 3$. This implies that the effect of the hub can be ignored for large γ. Thus, there exists competition between individuals, which hinders the consensus formation of the Axelrod model on SF networks with $\gamma > 3$.

From Eq. (5), we find that the disordered phases observed on the finite SF networks with $\gamma \leq 3$ are metastable. When SFPRs are repeatedly applied to these disordered phases, the perturbations play the role of random fluctuations which drive the metastable phases into an ordered (stable) phase. As a result, the observed disordered phases on finite SF networks with $\gamma \leq 3$ can easily evolve into an ordered phase when SFPRs are repeatedly applied. Moreover, if the system is in the ordered phase then the avalanches with finite s drive the system back to the unperturbed configuration. On the other hand, since the given value of Q ensures that the Axelrod model on 2D square lattices or ER networks is in a disordered phase when $N \to \infty$, infinite avalanches are needed to drive a disordered phase into an ordered phase. However, such huge avalanches hardly occur on 2D square lattices or ER networks as shown in Figs. 1 and 2. Therefore, on 2D lattices or ER networks the SFPRs cannot drive the system into an ordered phase in the limit $N \to \infty$.

4. Summary and discussion

We study the effect of the underlying topologies on the spreading of single feature perturbation in the Axelrod model. From numerical simulations, we find that the distribution of avalanche sizes induced by the repeated action of SFPRs satisfies the FSS ansatz (1)–(3) on 2D lattices and on random networks. However, on SF networks with $\gamma \leq 3$ we find that $P(s, N)$ does not depend on N. The results imply that only when the underlying topologies are highly heterogeneous ($\gamma \leq 3$) the repeated action of SFPRs can drive the disordered configuration of the Axelrod model into an ordered one.

The origin of the different behavior in $P(s, N)$ can be understood by the relationship between the underlying topology and Q_c. As mentioned in Fig. 5, there is no phase transition and the system is always in the ordered phase, when $\gamma \leq 3$ in the limit $N \to \infty$. Thus, the disordered phases observed in a finite system are unstable against the repeated action of SFPR, because the hub plays a crucial role in the consensus formation as a cultural leader. However, in homogeneous networks (or SF networks with $\gamma > 3$), there exists a finite Q_c and the disordered phase is stable against SFPR, because of the absence of a hub (or cultural leader) in the system. These results indicate that the cultural leader changes its features and then all agents follow it if the interaction topology is an SF network with $\gamma \leq 3$. Since the interaction topologies between each individual in many real social networks belong to SF networks with $\gamma < 3$ [16], the following example can provide an important insight into the understanding of dynamical properties. An interesting and practical example is the spreading of new technologies, such as the mobile phone. Since the recent development of mobile technology, the use of mobile phones has been gradually increasing and more than 90% of US citizens are now using mobile phone technology [27]. This clearly shows that mobile phone technology almost replaces land-line communication. In this example, the mobile phone and land-line are two equally-functional alternatives. Thus, it clearly shows that the repeated action of random changes between two equally-functional alternatives leads to the ordered phase on SF networks with $\gamma \leq 3$ because the interaction topology in social networks is known to be a SF network with $\gamma < 3$ [16].

In real society, the effect of random change in cultural traits has drawn long-lasting research interests among various branches of science. Another obvious example is language. Pronunciation, grammar, and spelling have been changed at random among the equally-functional alternatives. The evolution of babies' names is also an example of random changes in cultural traits [28]. Such random changes are usually called cultural drift [1] and have been proposed as an appropriate model for the way in which equally-functional elements come and go in society [29,30]. Similar random changes in genetic evolution were also observed, which is called Wright's genetic drift [31]. In physics, cultural drift has been rephrased as random noise [9]. In our study we show how the underlying topologies affect the spreading of a random change. Therefore, our study takes a first step in the direction of a systematic study on the spreading of cultural drift in more realistic systems.

Acknowledgments

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