Available online at www.sciencedirect.com

ScienceDirect

Physica A 387 (2008) 3033–3038

PHYSICA A

www.elsevier.com/locate/physa

Random walks and diameter of finite scale-free networks

Sungmin Lee, Soon-Hyung Yook*, Yup Kim

Department of Physics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Republic of Korea

Received 28 September 2007; received in revised form 21 January 2008
Available online 31 January 2008

Abstract

Dynamical scalings for the end-to-end distance R_{ee} and the number of distinct visited nodes N_v of random walks (RWs) on finite scale-free networks (SFNs) are studied numerically. $\langle R_{ee}\rangle$ shows the dynamical scaling behavior $\langle R_{ee}(\bar{\ell}, t)\rangle = \bar{\ell}^{\alpha}(\gamma, N)g(t/\bar{\ell}^{z})$, where $\bar{\ell}$ is the average minimum distance between all possible pairs of nodes in the network, N is the number of nodes, γ is the degree exponent of the SFN and t is the step number of RWs. Especially, $\langle R_{ee}(\bar{\ell}, t)\rangle$ in the limit $t \to \infty$ satisfies the relation $\langle R_{ee}\rangle \sim \bar{\ell}^{\alpha} \sim d^{\alpha}$, where d is the diameter of network with $d(\bar{\ell}) \simeq \ln N$ for $\gamma \geq 3$ or $d(\bar{\ell}) \simeq \ln\ln N$ for $\gamma < 3$. Based on the scaling relation $\langle R_{ee}\rangle$, we also find that the scaling behavior of the diameter of networks can be measured very efficiently by using RWs.

PACS: 05.40.Fb; 89.75.Fb; 89.75.Hc; 05.90.+m

Keywords: Complex networks; Random walks

For almost a decade there have been many studies on the topological properties of complex networks, since many structures of physically interacting systems are shown to form nontrivial complex structures [1,2]. In these studies much effort has been put to investigate the physical origin of complex networks [3]. It has been found that most of the real web-like systems share several prominent structural features with the small-world networks and the scale-free networks. The small-world networks [4] are characterized by high clustering and small average path length (APL) or diameter which is defined as the longest geodesic path between all possible pairs of nodes [1,3]. The **scale-free networks (SFNs)** whose degree distribution $P(k)$ satisfies a power law $P(k) \sim k^{-\gamma}$ also have the SW property [1,3]. It is well known [5] that random networks (RNs) [6] and small-world networks [4] have very small APL ($\bar{\ell}$) and diameter (d) which scale as $d \sim \bar{\ell} \sim \ln N$, where N is the number of nodes. Recently, Cohen and Havlin [7] analytically showed that

$$\bar{\ell}(N) \sim \begin{cases} \ln\ln N, & 2 < \gamma < 3 \\ \ln N/\ln\ln N, & \gamma = 3 \\ \ln N, & \gamma > 3. \end{cases} \tag{1}$$

* Corresponding author. Tel.: +82 2 961 0253; fax: +82 2 957 8408.
E-mail address: syook@khu.ac.kr (S.-H. Yook).

0378-4371/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2008.01.101

For measuring $\overline{\ell}$, the breath-first algorithm (BFA) [8] is mainly used. This algorithm is known to scale as $O(N^2)$. In this paper, we will show that random walk (RW) can provide more efficient method to find the scaling behavior of d like Eq. (1).

The early studies on complex networks mainly focused on their topological properties. Recently, the physical systems whose elements interact along the links in complex networks have drawn much attention. Furthermore a number of studies have focused on the effects of the underlying topologies on the dynamical properties of such systems. Many dynamical systems on complex networks show rich behaviors which are far from the mean-field expectations and affected by the underlying topology [9,10]. For example, the dynamical properties of RWs on complex networks have been shown to be closely related to the topology of underlying networks [11,12]. Especially, the average number of distinct visited sites and the average end-to-end distance of RWs on small-world networks are known to satisfy the scaling law $O(p, N, t) = O_{\text{sat}} F(p^2 t, pN)$ [12]. Here, p is a density of shortcut (or the probability that each node has an additional shortcut) and O_{sat} is a saturated quantity.

It is well known that the SFNs have heterogeneous structures in which nodes with anomalously large number of degrees and nodes with small degrees co-exist [1,2]. In SFNs, the dynamical properties of several systems are affected by the second moment of degree distribution $\langle k^2 \rangle$ [13–16]. It is interesting to study how the structural heterogeneity affects the scaling properties of RWs on changing the degree exponent γ. In this paper, we mainly investigate the dynamical scaling relation for the end-to-end distance R_{ee} of RWs on finite SFNs with various γ. From the scaling relation, it will be shown that RWs on SFNs can provide much more efficient method to measure the scaling behavior of the diameter of finite SFNs than BFA. For the complementary purpose we also study the scaling relation for the number of distinct visited nodes N_v.

To generate SFNs, we use the static model [17] with the average degree $\langle k \rangle = 4$. In the static model, the weight $w_i = i^{-\alpha}$ is assigned to each node i ($i = 1, \ldots, N$). Then we choose two different nodes (i, j) with probabilities $w_i / \sum_k w_k$ and $w_j / \sum_k w_k$, and add an edge between them. In this network γ is given by $\gamma = (1 + \alpha)/\alpha$. Thus, by adjusting α we can obtain various values of γ [17].

Initially a random walker is placed on a randomly chosen node, s, on the network. At the time step $t + 1$, the walker jumps to a randomly chosen node among the nearest neighbor nodes of the node where the walker is at t. The probability $P(i, t)$ to find the walker at node i at t thus follows the relation

$$P(i, t+1) = \sum_{j=1}^{N} \frac{A_{ij}}{k_j} P(j, t). \tag{2}$$

Here, A_{ij} is the adjacency matrix whose elements are $A_{ij} = 1(0)$ if two nodes i and j are connected (disconnected). All quantities are averaged over 100 network realizations and 1000 different initial positions of RWs for each network realization.

Before discussing the dynamical properties of RWs, we investigate the topological property of the static SFNs which we use. In complex networks, there are two characteristic distances, diameter d and APL $\overline{\ell}$. As we shall show later, these characteristic length scales are important in determining the scaling behavior of end-to-end distance of RWs. Even though the definition of $\overline{\ell}$ is different from that of d, some authors do not discriminate $\overline{\ell}$ from d. Thus, in order to verify the compatibility of $\overline{\ell}$ and d, we first measure d as a function of $\overline{\ell}$. As shown in Fig. 1(a) we find the linear relationship $d \propto \overline{\ell}$ for any γ. Thus the scaling behavior of d is the same with that of $\overline{\ell}$, and we will use d and $\overline{\ell}$ without any distinction hereafter. As mentioned in Eq. (1), the scaling behavior of $\overline{\ell}$ depends on γ of SFNs. For $\gamma = 3$, the scaling behavior of $\overline{\ell}$ becomes more complicated and it also depends on the existence of loops. In the looped Barabási–Albert (BA) networks [18], $\overline{\ell} \sim \ln N/(\ln \ln N)$ [19]. In contrast $\overline{\ell} \sim \ln N$ in the loopless BA networks [19]. To find the best scaling relation of $\overline{\ell}$ of the static SFNs with $\gamma = 3$ we directly measure $\overline{\ell}$ using the BFA method (see Fig. 1(b).) The data in Fig. 1(b) shows that $\overline{\ell}$ of SFNs with $\gamma = 3$ seems to scale both as $\overline{\ell} \sim \ln N/\ln \ln N$ and $\overline{\ell} \sim \ln N$ for small N. However, as N increases, $\overline{\ell}$ deviates from the relation $\overline{\ell} \sim \ln N/\ln \ln N$ (see inset of Fig. 2(b)). As pointed out in Ref. [7], $\gamma = 3$ is marginal. Thus, finding a correct scaling behavior for $\gamma = 3$ is difficult through numerical simulations. In our analysis we estimate the standard errors of slopes of assumed relations, $\overline{\ell} \sim \ln N$ and $\overline{\ell} \sim \ln N/\ln \ln N$. The estimated errors are 0.2 for $\overline{\ell} \sim \ln N/\ln \ln N$ and 0.06 when $\overline{\ell} \sim \ln N$. The systematic deviation of $\overline{\ell}$ from the analytic expectation (1) shows that $\overline{\ell}$ of the static SFN model [17] does not have $\ln \ln N$ correction which is obtained by the tree-approximation [7]. This reflects the fact that other topological properties such as degree–degree correlation should be considered in the derivation of $\overline{\ell}$. In Fig. 1(c) we show that

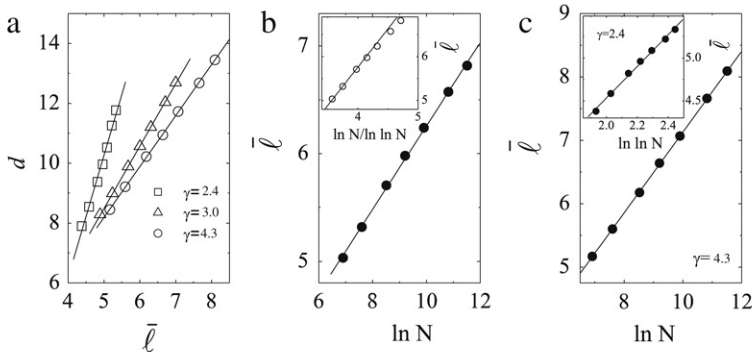

Fig. 1. (a) Plot of d against $\bar{\ell}$ for $N = 10^3 \sim 10^5$. The solid lines represent the relation $d \propto \bar{\ell}$. (b) and (c): Dependence of $\bar{\ell}$ on N for (b) $\gamma = 3$ and (c) $\gamma = 4.3$ and $\gamma = 2.4$ (inset of (c)). The solid lines represent the relations $\bar{\ell} \sim \ln N$ for $\gamma \geq 3$ and $\bar{\ell} \sim \ln \ln N$ for $\gamma = 2.4$ (inset of (c)). The inset of (b) shows that $\bar{\ell}$ for $\gamma = 3$ deviates from $\bar{\ell} \sim \ln N / \ln \ln N$ (solid line) when N increases.

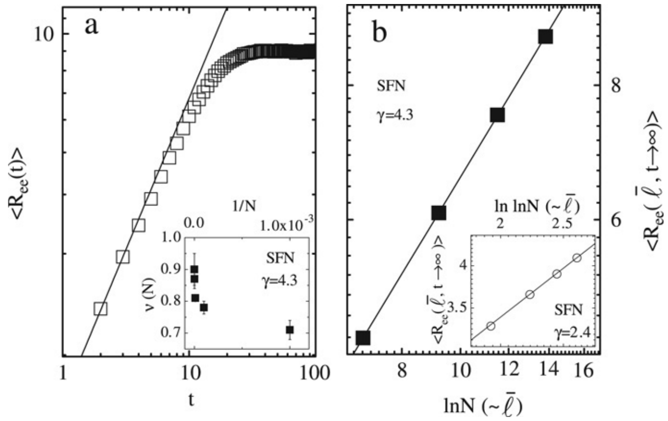

Fig. 2. (a) Plot of $\langle R_{ee} \rangle$ against t for $\gamma = 4.3$ when $N = 10^7$. The solid line corresponds to $\langle R_{ee} \rangle \sim t^{0.90}$. The inset displays the dependence of the measured ν on $1/N$. (b) Plot of $\langle R_{ee}(\bar{\ell}, t \to \infty) \rangle$ against $\ln N$ ($\sim \bar{\ell}$) for $\gamma = 4.3$ in log–log scale. The solid line corresponds to $\langle R_{ee}(\bar{\ell}, t \to \infty) \rangle \sim \bar{\ell}^{0.93}$. The inset shows $\langle R_{ee}(\bar{\ell}, t \to \infty) \rangle$ vs. $\ln \ln N$ ($\sim \bar{\ell}$) when $\gamma = 2.4$ in log–log scale. The solid line represents $\langle R_{ee}(\bar{\ell}, t \to \infty) \rangle \sim \bar{\ell}^{0.69}$.

$\bar{\ell}(N) \sim \ln N$ for $\gamma = 4.3$ and $\bar{\ell}(N) \sim \ln \ln N$ for $\gamma = 2.4$. We numerically confirm $\bar{\ell}(N) \sim \ln N$ for $\gamma \geq 3$ and $\bar{\ell}(N) \sim \ln \ln N$ for $2 < \gamma \lesssim 2.5$. In contrast we cannot rule out the possibility $\bar{\ell}(N) \sim \ln N$ for $2.6 \lesssim \gamma < 3$ from the numerical data. However we expect from Eq. (1) [7] that the relation $\bar{\ell}(N) \sim \ln \ln N$ recovers for $2.6 \lesssim \gamma < 3$ in the limit $N \to \infty$.

Now, let us discuss the average end-to-end distance $\langle R_{ee} \rangle$ of RW on SFNs. At each time step t, we measure the shortest distance, $R_{ee}(t)$, from the node where the random walker is to the node s. By averaging $R_{ee}(t)$ over different initial positions and network realizations we get $\langle R_{ee} \rangle$. Here the shortest distance between two nodes in networks means the shortest path length or the minimal number of steps between them.

A typical dependence of $\langle R_{ee} \rangle$ on t for finite SFNs is shown in Fig. 2(a). Fig. 2(a) shows the measured $\langle R_{ee} \rangle$ on SFNs with $\gamma = 4.3$ for $N = 10^7$. From the early-t behavior of $\langle R_{ee} \rangle$ (or the data for $t \lesssim 10$ in Fig. 2(a)) the obtained value of the exponent ν for the relation

$$\langle R_{ee}(t) \rangle \sim t^{\nu} \tag{3}$$

is $\nu \simeq 0.90(5)$. For other values of γ we obtain the same value, $\nu \simeq 0.90(5)$, when $N = 10^7$. The inset in Fig. 2(a) displays the dependence of the measured ν on the network size N, which indicates that the values of ν approach to 1 as N increases. The $\langle R_{ee}(t) \rangle$ does not increase indefinitely, but rapidly reaches a saturation value $\langle R_{ee}(\bar{\ell}, t \to \infty) \rangle$ after a very short crossover time τ_{ee}. Since τ_{ee} is nearly equal to 10 or slightly larger than 10 even for the very large network size (or $N = 10^7$), the expectation of $\nu = 1$ from the data for less than one decade of t or so seems to

Table 1
Estimates of the exponents α and z for SFNs with various γ's and random network (RN)

γ	α	z	γ	α	z
2.15	0.42(1)	0.46(3)	3.0	0.60(1)	0.66(4)
2.40	0.69(2)	0.76(3)	3.5	0.81(1)	0.90(4)
2.50	0.84(1)	0.93(4)	4.3	0.93(2)	1.03(5)
			5.7	0.98(4)	1.08(7)
			RN	1.00(5)	1.11(8)

z is calculated from the relation α/ν. The used value of ν is $\nu = 0.90(5)$ which is obtained from the network with $N = 10^7$.

be physically unsound. However, since an earlier analytical study on the walks of a Cayley tree also suggests such $\nu = 1$ behavior [20], $\nu \to 1$ as $N \to \infty$ is a physically more plausible one. We thus expect that $\nu \to 1$ as $N \to \infty$ regardless of γ.

In Fig. 2(b) we display the $\langle R_{ee}(\bar{\ell}, t \to \infty)\rangle$ for $\gamma = 4.3$ as a function of $\bar{\ell}$ with $\bar{\ell} \sim \ln N$. As shown in Fig. 2(b), $\langle R_{ee}(\bar{\ell}, t \to \infty)\rangle$ for $\gamma = 4.3$ satisfies the power-law:

$$\langle R_{ee}(\bar{\ell}, t \to \infty)\rangle \sim \bar{\ell}^{\alpha}, \tag{4}$$

with $\alpha = 0.93(2)$. $\langle R_{ee}(\bar{\ell}, t \to \infty)\rangle$ for various γ is also found to satisfy the power law (4) very well. The obtained α's for various γ by assuming $\bar{\ell} \sim \ln\ln N$ for $\gamma \leq 2.5$ and $\bar{\ell} \sim \ln N$ for $\gamma \geq 3$ are displayed in Table 1. For $2.6 \lesssim \gamma < 3$, we find that $\langle R_{ee}\rangle$ scales both as $(\ln N)^{\alpha}$ with $\alpha < 1$ and $(\ln\ln N)^{\alpha}$ with $\alpha > 1$. For example, we obtain $\alpha = 0.30(1)$ by use of the relation $\bar{\ell} \sim \ln N$ for $\gamma = 2.7$. In contrast $\alpha = 1.49(1)$ is obtained using $\bar{\ell} \sim \ln\ln N$ for $\gamma = 2.7$. By considering that the probability to find a random walker at a node of degree k is $p_v(k) \sim k$ [11], $\langle R_{ee}(\bar{\ell}, t \to \infty)\rangle < d$ and α should not be greater than 1. Thus, only $\langle R_{ee}(\bar{\ell}, t \to \infty)\rangle \sim (\ln N)^{\alpha}$ leads the correct physical expectation $\alpha \leq 1$ for $2.6 \lesssim \gamma < 3$. This result is consistent with the direct numerical measurement of $\bar{\ell}$. Thus it is very difficult to numerically see $\langle R_{ee}\rangle \simeq (\ln\ln N)^{\alpha}$ with $\alpha < 1$ for $2.6 \lesssim \gamma < 3$, since we have checked on the network with the size up to $N = 10^6$. However, relying on the analytic arguments such as Ref. [7] we expect that the scaling relation $\langle R_{ee}\rangle \sim (\ln\ln N)^{\alpha}$ with $\alpha < 1$ would be recovered even for $2.6 \lesssim \gamma < 3$ in the limit $N \to \infty$. Moreover, the crossover of the functional form of $\bar{\ell}(N)$ causes the non-monotonic change of α.

From Eqs. (3) and (4), the dynamical scaling relation

$$\langle R_{ee}(\bar{\ell}, t)\rangle = \bar{\ell}^{\alpha}(\gamma, N) g(t/\bar{\ell}^{z}) \tag{5}$$

is expected. The scaling function $g(x)$ then satisfies the relation

$$g(x) \sim \begin{cases} x^{\nu}, & x \ll 1 \\ \text{const.}, & x \gg 1. \end{cases} \tag{6}$$

The dynamical scaling relation (5) also physically means that τ_{ee} scales as $\tau_{ee} \sim \bar{\ell}^{z}$. The dynamic exponent z is evaluated from the measured α and ν through the relation $z = \alpha/\nu$ and displayed in Table 1. To calculate z we use $\nu = 0.90(5)$ measured on the SFN with $N = 10^7$. The values of α's listed in Table 1 agree with the theoretical expectation that $\alpha \leq 1$. Since $\nu \to 1$ as $N \to \infty$, the asymptotic value of z is expected to be the same with α. However, the finite-size effect causes $\nu < 1$. As a result, the values of z listed in Table 1 slightly deviate from the asymptotic values for all γ, but are consistent with the expected asymptotic values considering the error estimates. Fig. 3 shows the scaling plot of $\langle R_{ee}\rangle$ measured on SFNs with $\gamma = 2.4, 3.0, 4.3$ and on random networks (RNs) for $N = 10^3, 10^4, 10^5$ and 10^6 using Eq. (5). As shown in Fig. 3, $\langle R_{ee}\rangle$'s for various N collapse very well into a single scaling curve with the exponents listed in Table 1 for each network topology.

Eqs. (3)–(5) provide another interesting way to find the *scaling behavior* of $\bar{\ell}$ in SFN. The results imply that the computing time needed for the measurement of scaling behavior of $\bar{\ell}$ by RW method increases as $O((\ln N)^{z})$ or $O((\ln\ln N)^{z})$. Since the computing time to measure $\bar{\ell}$ by BFA increases as $O(N^2)$, the RW method is far more efficient to find the scaling behavior of $\bar{\ell}$.

For the sake of comparison we now explain the average number of distinct visited sites $N_v(t)$ on SFNs with various γ. Fig. 4 shows the scaling plot of $N_v(t)$ for $\gamma = 4.3$ and 2.4 against t/N for $N = 10^3, 10^4, 10^5$ and 10^6. $N_v(t)$ on

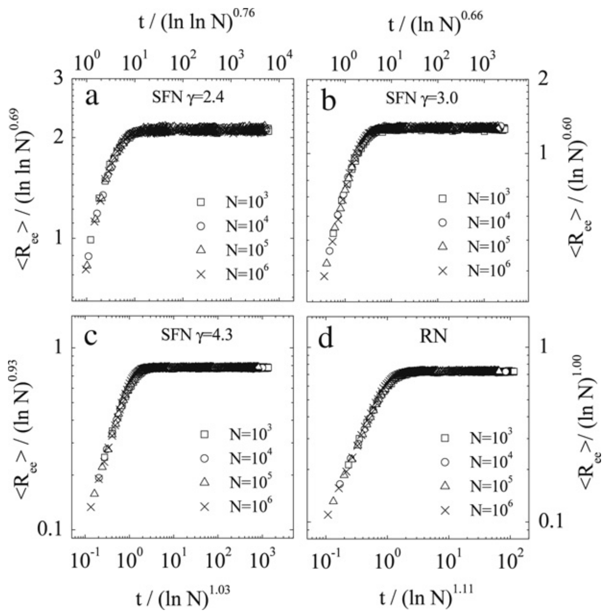

Fig. 3. Scaling plot of $\langle R_{ee} \rangle$ on SFNs with $\gamma = 2.4$ (a), 3.0 (b), 4.3 (c) and on RNs (d) for $N = 10^3$, 10^4, 10^5 and 10^6.

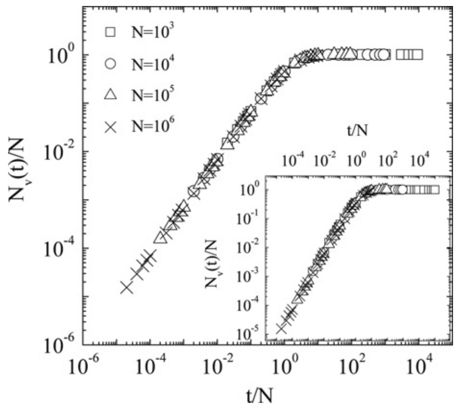

Fig. 4. This figure shows the scaling collapse for N_v on SFN with $\gamma = 4.3$ and 2.4 (inset).

SFN satisfies the scaling relation

$$N_v = Nf(t/N) \tag{7}$$

with

$$f(x) \sim \begin{cases} x, & x \ll 1 \\ 1, & x \gg 1. \end{cases} \tag{8}$$

We have checked that the scaling relation (7) holds very well for SFNs with any γ.

The scaling relation (7) is slightly different from that of the regular lattice and the small-world network. On the infinite D-dimensional lattices N_v depends on D. In the limit of $t \to \infty$ [21], $N_v \sim \sqrt{t}$ in $D = 1$, $N_v \sim t/\ln t$ in $D = 2$ and $N_v \sim t$ for $D > 2$. On small-world networks, Almaas et al. [12] showed that the scaling behavior of N_v crossovers from the $N_v \sim \sqrt{t}$ to $N_v \sim t$ as t increases. This means that if the random walker does not reach the shortcut, then the walker always sees the regular lattice structure ($D = 1$). On the other hand, if the walker meets the shortcut, then the behavior of walker follows the mean-field result and finally N_v saturates to N due to the finite-size of the network. However, each node in SFN can be connected to any other nodes and SFN can be regarded as an infinite-dimensional structure. Thus, N_v on SFN follows the mean-field behavior after the first few steps. From Eqs. (7) and (8) the time $t_\times$ at which $N_v = N$ is given by $t_\times \simeq N$. This implies that the walker can sample a new region at each time step by using shortcuts until it visits all nodes in the network. Thus the statistical properties such as the average of certain quantity can be significantly enhanced even for small RW steps.

Moreover, since $\langle R_{ee}(t)\rangle \sim t^{1/2}$ and $\langle N_v(t)\rangle \sim t$ for $D \geq 2$, we can obtain a relation between the N_v and the number of accessible nodes (or volume) N_{ac} within the radius $\langle R_{ee}(t)\rangle$ for $D \geq 2$:

$$\frac{N_v}{N_{ac}} \sim t^{1-D/2}. \tag{9}$$

Thus, N_{ac}/N_v diverges as $t \to \infty$ which indicates the *transient* behavior of RWs [22]. The transient behavior becomes much more striking in the limit $D \to \infty$. From $\tau_{ee} \sim (\ln N)^z$ (or $(\ln\ln N)^z$) and $t_\times \simeq N$, we know $\tau_{ee} \ll t_\times$ on SFN. This result means that the walker effectively moves to the end of SFN without visiting all nodes. As a result, we can measure the scaling behavior of $d(N)$ in much shorter time than BFA algorithm (for example, see Figs. 1 and 3).

We now briefly summarize our results. We investigate the scaling properties of RWs on SFNs. We measure the end-to-end distance R_{ee} and the number of distinct visited sites N_v. From the scaling ansatz for $\langle R_{ee}\rangle$ as Eq. (5), we have measured the scaling exponents α, ν and z for $\langle R_{ee}\rangle$ on various networks. From the scaling relation we find the dependency of $\bar{\ell}$ on γ and N. Based on the simple scaling arguments, we also discuss the theoretical reasons why scaling behavior of $d(N)$ of networks can be measured far more efficiently by RW method than by BFA. Finally, Eq. (4) can be easily recovered for $\gamma > 3$ by assuming $r_{k,k'} \propto N^\delta \left(kk'\right)^q \left(\ln(kk')\right)^\alpha$ with $\delta = (-2q + 2\gamma - 1)/(\gamma - 1)$. Here $r_{k,k'}$ is the average minimal distance between nodes of degree k and k'.

Acknowledgements

This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MOST) (No. R01-2007-000-10910-0, No. R01-2006-000-10470-0, and F01-2006-000-10093-0).

References

[1] S.N. Dorogovtsev, J.F.F. Mendes, Adv. Phys. 51 (2002) 1079;
R. Albert, A.-L. Barabási, Rev. Modern Phys. 74 (2002) 47.
[2] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, A.-L. Barabási, Nature 407, 651 (2001);
S.-H. Yook, Z.N. Oltvai, A.-L. Barabási, Porteomics 4 (2004) 928.
[3] S.N. Dprogovtsev, J.F.F. Mendes, Evolution of Networks, Oxford University Press, Cambridge, UK, 2003.
[4] D.J. Watts, S.H. Strogatz, Nature (London) 393 (1998) 440.
[5] E. Ben-Naim, P.L. Krapivsky, S.N. Majumdar, Phys. Rev. E 64 (2001) R035101.
[6] P. Erdös, A. Rényi, Publ. Math. 6 (1959) 290.
[7] R. Cohen, S. Havlin, Phys. Rev. Lett. 90 (2003) 058701.
[8] R. Sedgwick, Algorithms, Addison-Wesley, 1988;
M.E.J. Newman, Phys. Rev. E 64 (2001) 016132.
[9] A.V. Goltsev, S.N. Dorogovtsev, J.F.F. Mendes, Phys. Rev. E 67 (2003) 026123.
[10] R. Cohen, D. ben-Avraham, S. Havlin, Phys. Rev. E 66 (2002) 036113.
[11] J.D. Noh, H. Rieger, Phys. Rev. Lett. 92 (2004) 118701.
[12] E. Almaas, R.V. Kulkarni, D. Stroud, Phys. Rev. E 68 (2003) 056105.
[13] S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes, Phys. Rev. Lett. 66 (2002) 016104.
[14] S. Lee, Y. Kim, Phys. Rev. E 71 (2005) 057102.
[15] S. Kwon, S. Lee, Y. Kim, Phys. Rev. E 73 (2006) 056102.
[16] S. Lee, S.-H. Yook, Y. Kim, Phys. Rev. E 74 (2006) 046118.
[17] K.-I. Goh, B. Kahng, D. Kim, Phys. Rev. Lett. 87 (2001) 278701.
[18] A.-L. Barabási, R. Albert, Science 286 (1999) 509.
[19] G. Szabó, M. Alava, J. Kertész, Phys. Rev. E 66 (2002) 026101.
[20] Y. Kim, Phys. Rev. A 39 (1989) 4923.
[21] A. Dvoretzky, P.P. Erdös, Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, California, 1950, pp. 353–367.
[22] G. Pólya, Mathematische Annalen 84 (1921) 149.