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Agglomerative percolation on the Bethe lattice and the triangular cactus

Huiseung Chae, Soon-Hyung Yook and Yup Kim

Department of Physics and Research Institute for Basic Sciences, Kyung Hee University,
Seoul 130-701, Korea

E-mail: ykim@khu.ac.kr

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Abstract

Agglomerative percolation (AP) on the Bethe lattice and the triangular cactus is studied to establish the exact mean-field theory for AP. Using the self-consistent simulation method based on the exact self-consistent equations, the order parameter P_∞ and the average cluster size S are measured. From the measured P_∞ and S , the critical exponents β_k and γ_k for $k = 2$ and 3 are evaluated. Here, β_k and γ_k are the critical exponents for P_∞ and S when the growth of clusters spontaneously breaks the Z_k symmetry of the k -partite graph. The obtained values are $\beta_2 = 1.79(3)$, $\gamma_2 = 0.88(1)$, $\beta_3 = 1.35(5)$ and $\gamma_3 = 0.94(2)$. By comparing these exponents with those for ordinary percolation ($\beta_\infty = 1$ and $\gamma_\infty = 1$), we also find $\beta_\infty < \beta_3 < \beta_2$ and $\gamma_\infty > \gamma_3 > \gamma_2$. These results quantitatively verify the conjecture that the AP model belongs to a new universality class if the Z_k symmetry is broken spontaneously, and the new universality class depends on k .

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(Some figures may appear in colour only in the online journal)

1. Introduction

Percolation transition describes the emergence of large-scale connectivity [1]. It has been extensively studied in various branches of science due to its wide range of applications to many phenomena such as polymerization, resistor networks and epidemic spreading [1]. The first theoretical model for percolation was random or ordinary percolation in which a vacant site or a vacant bond of a background lattice is randomly chosen to be occupied. The percolation transition in random percolation is normally known to be continuous [1]. Percolation has been extensively studied during the last three or four decades to be considered as a mature branch of sciences.

However, the anomalous physical properties of exotic percolation models recently triggered some new studies. One kind of study [2] was on explosive percolation. Explosive

percolation was first argued to show discontinuous transition on the complete graph [2, 3]. But subsequent studies have shown that the transition of explosive percolation on the complete graph (or the mean-field transition) is continuous [4–8].

Another kind of study was on agglomerative percolation (AP) [9–13]. In AP one cluster is randomly selected instead of a bond or a site. Then the selected cluster merges all the nearest-neighboring clusters to form a new cluster. Using analytical methods and numerical simulations, APs on a one-dimensional ring [9], two-dimensional lattices [10], critical trees [11] and complex networks [12] were studied. The phase transition in AP was shown to be continuous. However, AP was proved to belong to a new universality class different from that of the random percolation if the base structure is bipartite [13]. In AP on a bipartite structure like a two-dimensional square lattice, the merging process spontaneously breaks the Z_2 symmetry at the transition threshold, which is the origin of the new universality class [13]. In contrast, the universality class of AP on a triangular lattice, which is not bipartite, was shown to be the same as that of random percolation [13]. Through these studies, the AP on a bipartite graph was shown to belong to a new universality class different from that of random percolation on the same graph.

To understand the critical phenomena of a new model clearly and precisely, the exact mean-field theory (MFT) for the model must be first understood. However, MFT for AP on a bipartite graph was not clearly understood, yet. To obtain the MFT of AP, a generating function approach to AP on the Erdős–Rényi (ER) random network was attempted [12]. This analytic approach predicted the critical exponent γ as $\gamma = 1/2$ [12]. But from the numerical simulation on the ER graph, $\gamma = 0.88(10)$ was obtained, which is significantly larger than $\gamma = 1/2$. Furthermore, the ER graph is not exactly bipartite. The numerical simulation study on the exact bipartite random graph earned only the critical exponent ν and the fractal dimension D of the giant cluster as $\nu = 4.7(2)$ and $D = 0.567(6)$, which are close to those for AP on ER network but differ by more than one standard deviation [13]. Therefore, at this stage, MFT for AP on a bipartite graph is far from completion.

Recently, the complete graph has widely been used as a testbed for MFT [2, 4, 6, 7]. However, the complete graph is not bipartite and one growth step of AP on the graph makes the entire graph a new single cluster. In contrast, the Bethe lattice (infinite homogeneous Cayley tree) on which AP can be well defined is bipartite. Moreover, the Bethe lattice is physically a very important substrate or medium on which MFTs for various physical models become exact [14]. The analytic treatments of magnetic models [15], percolation [1, 14], localization [14] and diffusion [16] on the Bethe lattice give important physical insights into the subsequent developments of the corresponding research fields. Therefore, if AP on the Bethe lattice is completely understood, one knows MFT for AP exactly.

One of the theoretical merits of the Bethe lattice is that one can set up exact self-consistent equations on the lattice. Recently, we have developed an exact self-consistent simulation method for the arbitrary percolation process on the Bethe lattice [8]. From the self-consistent simulation method, we have shown that the Achlioptas-type explosive percolation [2] undergoes continuous transition [8]. In this paper, the critical properties of AP on the Bethe lattice are studied by the use of the developed self-consistent simulation. By the self-consistent simulation, the order parameter P_∞ and the average size S of finite clusters on the Bethe lattice are directly measured. Therefore, the exponents β and γ for P_∞ and S are also obtained directly without using the conventional finite size scaling theory, and our work can indeed establish the exact MFT of AP.

In addition, Lau *et al* suggested the modified AP model in which the growth of clusters spontaneously breaks the Z_k symmetry of a k -partite graph. From now on we call the modified model AP_k [13]. So AP_2 means the original AP in which the growth of clusters breaks the Z_2

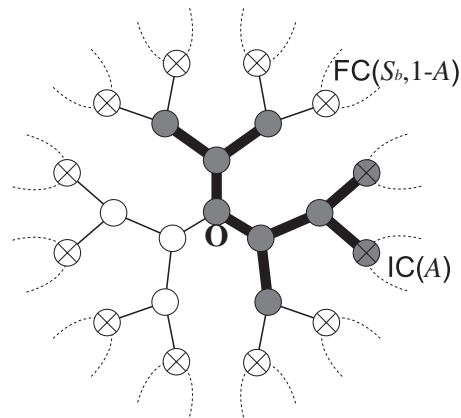


Figure 1. Schematic diagram for the arbitrary percolation process on the Bethe lattice with $z = 3$. The center part consists of a three-generation Cayley tree from O with edge sites denoted by \otimes . Each edge site is connected to an infinite cluster (IC) with the probability A or to a finite cluster (FC) of average size S_b with the probability $1 - A$. Thick lines mean occupied bonds and thin lines mean vacant bonds. The cluster containing O will be an infinite cluster with probability $1 - (1 - A)^2$.

symmetry of a bipartite graph. Based on a simple argument, the transition of AP_k is conjectured to belong to a new universality class depending on k [13]. However, the conjecture has never been confirmed quantitatively, yet. Therefore, in this paper we also study the MFT of the AP_3 model on the triangular cactus, which is an expanded structure of the Bethe lattice and exactly tripartite [14, 17] (see section 4). Since the Bethe lattice has a tree structure, like a critical tree [11], the Bethe lattice is k -partite (tripartite, four-partite, etc) as well as bipartite. So a percolation process on the Bethe lattice (or on any tree structure) which breaks a particular Z_k symmetry belongs to a new universality class depending on k as AP_2 on the Bethe lattice gives the mean-field bipartite universality class. Since the MFT of AP_3 is first studied quantitatively by the use of AP_3 on the triangular cactus, AP_3 on the Bethe lattice is also studied to confirm the result for AP_3 on the triangular cactus. By the self-consistent simulations the mean-field exponents β and γ for AP_3 , or β_3 and γ_3 , are obtained.

Finally from the results for AP_2 on the Bethe lattice and AP_3 on the triangular cactus and the Bethe lattice, relations among the critical exponents of AP_k 's are provided in the mean-field level.

This paper is organized as follows. In section 2, the self-consistent simulation method on the Bethe lattice is explained. The results for the critical phenomena of the ordinary AP or AP_2 obtained from the self-consistent simulation are shown in section 3. AP_3 on both the triangular cactus and the Bethe lattice is defined and studied in section 4. Finally, we summarize our results in section 5.

2. Self-consistent simulation

The Bethe lattice with the coordination number z is the infinite Cayley tree in which z identical infinite branches of tree structures are connected to the center site O as schematically shown in figure 1 [1, 14]. The essential difference of the Bethe lattice from the random infinite tree is the symmetric structure of identical trees centering on O . Let us first set up self-consistent equations for arbitrary percolation on the Bethe lattice [8]. To set up self-consistent equations on the Bethe lattice, the equations must be considered by focusing on O because of the

symmetric structure. First consider a part of the Bethe lattice (a finite Cayley tree) with m generations from \mathbf{O} , which has in total $N_m = 1 + z(k^m - 1)/(k - 1)$ sites, where $k \equiv z - 1$. To make a complete Bethe lattice, one should add an infinite branch to each of zk^{m-1} edge sites.

Now we first calculate the order parameter $P_\infty(p)$ at a given fraction p of the occupied bonds, which is defined as the probability that \mathbf{O} is connected to infinity by occupied bonds. Theoretically, $P_\infty(p)$ depends on two physical variables as shown in figure 1. One is the probability A that an edge site is connected to infinity through an infinite cluster on the branch connected to the edge site. The other is the average size S_b of finite clusters on the branch connected to the edge site. If A is given *a priori*, then one calculates the probability that \mathbf{O} belongs to an infinite cluster in a given configuration. For example, in the configuration of figure 1, \mathbf{O} belongs to an infinite cluster with probability $(1 - (1 - A)^2)$, because \mathbf{O} is connected to two edge sites through a finite cluster. Similarly, if \mathbf{O} is connected to t edge sites through a finite cluster within the m -generation tree, \mathbf{O} is connected to infinity with probability $(1 - (1 - A)^t)$. Let us call a finite cluster on the m -generation tree which contains \mathbf{O} and has s sites and t edge sites an st -cluster on the m -generation tree. The existence probability of an st -cluster on the m -generation tree generally depends on p, A and S_b as $P_{mst}(p, A, S_b)$. Then the order parameter, $P_{\infty m}(p, A, S_b)$, which is obtained by using the Bethe lattice composed of the m -generation tree and zk^{m-1} infinite branches is physically the same as the P_∞ , i.e., $P_\infty = P_{\infty m}(p, A, S_b)$. Thus,

$$P_\infty(p) = P_{\infty m}(p, A, S_b) \equiv 1 - \sum_{t=0}^{zk^{m-1}} (1 - A)^t \sum_{s=1}^{N_m} P_{mst}(p, A, S_b). \quad (1)$$

Since (1) holds for arbitrary m , a self-consistent equation for $P_\infty(p)$,

$$P_\infty(p) = P_{\infty m}(p, A, S_b) = P_{\infty m'}(p, A, S_b), \quad (2)$$

holds for any combination of $\{m, m'\}$.

One can also obtain a self-consistent equation for the average size, $S(p)$, of the finite clusters containing \mathbf{O} from the Bethe lattice composed of the m -generation tree and zk^{m-1} infinite branches. In the configuration of figure 1, the cluster which contains \mathbf{O} has nine sites ($s = 9$) and two edge sites ($t = 2$) within the three-generation tree. If the cluster is finite, the size of the cluster is $9 + 2S_b$. Similarly, $S(p)$ is written as

$$S(p) = S_m(p, A, S_b) \equiv \frac{\sum_{t,s} (1 - A)^t P_{mst}(p, A, S_b) [s + tS_b]}{1 - P_\infty}, \quad (3)$$

because a cluster with s sites and t edge sites within the m -generation tree becomes a finite cluster with probability $(1 - A)^t$. Here, $1 - P_\infty = \sum_{t=0}^{zk^{m-1}} (1 - A)^t \sum_{s=1}^{N_m} P_{mst}(p, A, S_b)$. Then, a self-consistent equation for $S(p)$,

$$S(p) = S_m(p, A, S_b) = S_{m'}(p, A, S_b), \quad (4)$$

holds for any combination of $\{m, m'\}$. From equations (2) and (4), one can reproduce the well-known self-consistent equations for the random (or ordinary) bond percolation on the Bethe lattice [1]. For $m' = 1$, $P_{1st}(p, A, S_b)$ for the random percolation is written as $P_{1st}(p, A, S_b) = P_{1st}(p) = \binom{z}{t} p^t (1 - p)^{z-t} \delta_{s,t+1}$. Similarly, for $m = 2$, $P_{2st}(p) = \sum_{l=0}^z \binom{z}{l} \binom{kl}{t} \delta_{(l+t+1),s} p^{s-1} (1 - p)^{k(s-t)-s+2}$. By applying these exact formulas to (1) and (2), one can derive a self-consistent equation for A on the Bethe lattice:

$$A = 1 - (1 - pA)^k. \quad (5)$$

Comparing (5) with the random bond percolation, one can readily obtain the conventional self-consistent equation on the Bethe lattice as $R = (1 - p + pR)^k$, with $R \equiv 1 - A$ [1, 8, 14].

In addition, from (3) and (4), one can easily derive the conventional mean-field results of S_b for the random percolation as

$$S_b = \frac{kp(1 - A)}{1 - kp + (k - 1)pA} \tag{6}$$

and

$$S(p) = \frac{1 + p - 2pA}{1 - kp + (k - 1)pA}, \tag{7}$$

where A is the root of (5) [1, 8, 14].

Any percolation process can be treated analytically by using (2) and (4), if the exact mathematical form of $P_{mst}(p, A, S_b)$ for the percolation is known. However, the exact form of $P_{mst}(p, A, S_b)$ for arbitrary percolation such as AP and explosive percolation is very difficult to obtain, whereas the form for random percolation is well known [1]. Therefore, to study a percolation process in which the exact form for $P_{mst}(p, A, S_b)$ cannot be obtained, $P_{mst}(p, A, S_b)$ should be estimated numerically. One such numerical method is a simulation method, which we called the self-consistent simulation [8]. One first measures the number, $N_{mst}(p, A, S_b)$, of st -clusters on the m -generation tree occurring in the simulation. Then, $P_{mst}(p, A, S_b)$ for a given percolation process is estimated by $P_{mst}(p, A, S_b) = N_{mst}(p, A, S_b)/N_{\text{cluster}}$, where $N_{\text{cluster}} = \sum_{st} N_{mst}(p, A, S_b)$.

In some percolation models, one cannot perform simulations without knowing A and S_b . If A and S_b are not known *a priori*, the iteration of a simulation process is needed in the self-consistent simulation. The iteration starts with guessed values for $A^{(1)}$ and $S_b^{(1)}$. Then the following simulation process is iterated. In the i th simulation process $P_{mst}(p, A^{(i)}, S_b^{(i)})$ is estimated by simulations using the known values of $A^{(i)}$ and $S_b^{(i)}$. Then $A^{(i+1)}$ and $S_b^{(i+1)}$ are obtained by numerically solving the equations

$$\sum_{t=0}^{zk^{m-1}} (1 - A^{(i+1)})^t \sum_{s=1}^{N_m} P_{mst}(p, A^{(i)}, S_b^{(i)}) = \sum_{t=0}^{zk^{m'-1}} (1 - A^{(i+1)})^t \sum_{s=1}^{N_{m'}} P_{m'st}(p, A^{(i)}, S_b^{(i)}). \tag{8}$$

and

$$\begin{aligned} & \sum_{t,s} (1 - A^{(i+1)})^t P_{mst}(p, A^{(i)}, S_b^{(i)}) [s + tS_b^{(i+1)}] \\ &= \sum_{t,s} (1 - A^{(i+1)})^t P_{m'st}(p, A^{(i)}, S_b^{(i)}) [s + tS_b^{(i+1)}]. \end{aligned} \tag{9}$$

Here (8) and (9) come from (1), (2), (3) and (4). Then from $A^{(i+1)}$ and $S_b^{(i+1)}$, the next $(i + 1)$ th simulation process is carried out. This simulation process is repeated until A and S_b reach the saturation values or until both equations $A^{(i+1)} = A^{(i)}$ and $S_b^{(i+1)} = S_b^{(i)}$ hold. By applying the saturated values of A and S_b to (1) and (3), P_∞ and S can be calculated at given p . In this paper, $P_{mst}(p, A^{(i)}, S_b^{(i)})$ and other relevant quantities are estimated by averaging over at least 10^6 simulation runs.

In the self-consistent simulation, we should be careful to choose m' ($< m$) for a given m as addressed in [8]. If m' is too small, then the clusters within the m' -generation tree cannot convey the physical properties of the corresponding percolation enough to give physically plausible solutions for the self-consistent equations (2) and (4). If m' is very close to m , $P_{\infty m'}(p, A, S_b)$ is numerically not so much distinct from $P_{\infty m}(p, A, S_b)$ and the self-consistent equations (2) and (4) hardly give the physically right solution. From the simulations with various sets of $\{m, m'\}$, it is confirmed that a suitable choice of m' should be in the interval $m/3 < m' \leq m/2$ as shown in figures 2, 3 and 5–8.

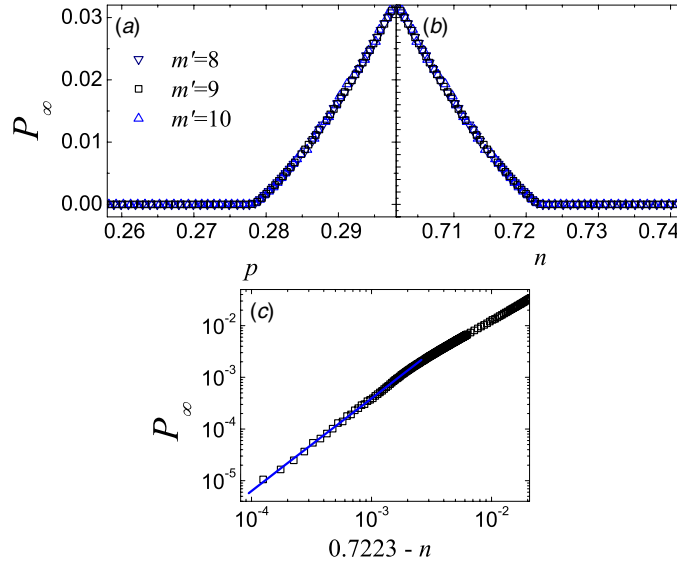


Figure 2. Order parameter P_∞ for AP_2 from the self-consistent simulation on the Bethe lattice with $z = 3$ using the m -generation tree and m' -generation tree. (a) Plots of P_∞ against p for the combinations $\{m = 20, m' = 8\}$ (∇), $\{m = 20, m' = 9\}$ (\square) and $\{m = 20, m' = 10\}$ (\triangle). (b) Plots of P_∞ against n for the same combinations as in (a). (c) Plots of P_∞ against $n_c - n$ with $n_c = 0.7223$ for the combination $\{m = 20, m' = 9\}$. The line denotes the relation $P_\infty \simeq (n_c - n)^{\beta_2}$, with $n_c = 0.7223$ and $\beta_2 = 1.79$.

3. AP_2 model

In the ordinary AP or AP_2 [10, 13], one cluster is randomly selected instead of a bond or a site and the selected cluster merges all the nearest-neighboring clusters to form a new cluster. This means that in each growth process multiple bonds can be occupied at the same time. Therefore, the natural control parameter for AP_2 is the number n of clusters per site instead of the fraction p of occupied bonds [10, 13]. In a tree structure, like the m -generation Cayley tree, n linearly depends on p as $n = 1 - p + p/N_m$. Thus, one can easily analyze the percolation transition by use of n if the dependence on p is known. In this sense, the results of the self-consistent simulation are thus analyzed by the use of n even though the results for both p and n are displayed in figures 2 and 3.

The results of the self-consistent simulation for AP_2 on the Bethe lattice with $z = 3$ using the m -generation tree and m' -generation tree are displayed in figures 2 and 3. The combinations of $\{m, m'\}$ used for the simulations are $\{m = 20, m' = 8\}$, $\{m = 20, m' = 9\}$ and $\{m = 20, m' = 10\}$. As emphasized in section 2, the simulation results are nearly identical for various m' ($= 8, 9, 10$) (see figures 2(a), (b) and 3(a), (b)).

To obtain the critical density n_c and the order parameter exponent β_2 , P_∞ for $\{m = 20, m' = 9\}$ in figure 2(c) is analyzed based on the equation

$$P_\infty \simeq (n_c - n)^{\beta_2}, \tag{10}$$

which holds for the ordered phase or for $n < n_c$ near the critical point, i.e., $n \rightarrow n_c^-$. The obtained n_c and β_2 are $n_c = 0.7223(1)$ and $\beta_2 = 1.79(3)$. We also obtain nearly the same n_c and β_2 within the estimated error when P_∞ 's for $m' = 10$ and $m' = 8$ in figure 2(b) are analyzed. For another consistent check the self-consistent simulation on the Bethe lattice with

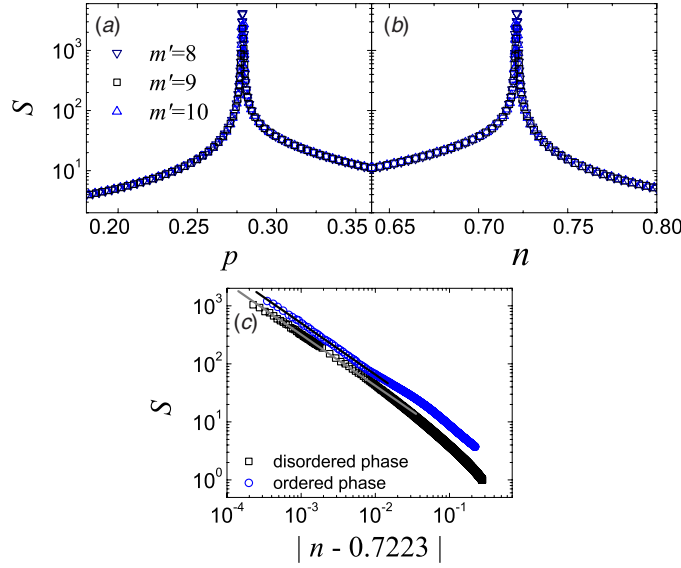


Figure 3. Average size S of finite clusters for AP_2 from the self-consistent simulation on the Bethe lattice with $z = 3$ using the m -generation tree and m' -generation tree. (a) Plots of S against p for the combinations $\{m = 20, m' = 8\}$ (∇), $\{m = 20, m' = 9\}$ (\square) and $\{m = 20, m' = 10\}$ (\triangle). (b) Plots of S against n for the same combinations as in (a). (c) Plots of S against $|n_c - n|$ for the ordered phase ($n < n_c$, \circ) and disordered phase ($n > n_c$, \square) with $n_c = 0.7223$ for the combination $\{m = 20, m' = 9\}$. The lines denote the relations $S \simeq |n - n_c|^{-\gamma_2^-}$ for $n > n_c$ and $S \simeq |n - n_c|^{-\gamma_2^+}$ for $n < n_c$ with $\gamma_2^- = \gamma_2^+ = 0.88$ and $n_c = 0.7223$.

$z = 6$ is carried out to obtain $\beta_2 = 1.79(3)$. These numerical results for β_2 are close to the previous estimate $\beta_2 = 1.78(8)$ on the ER graph [12], but our estimate has much smaller error.

From the data for the average size S of finite clusters and the equation

$$S \simeq \begin{cases} |n - n_c|^{-\gamma_2^-} & \text{if } n_c < n \\ |n - n_c|^{-\gamma_2^+} & \text{if } n_c > n \end{cases}, \quad (11)$$

we also estimate n_c , γ_2^- and γ_2^+ . S 's in figure 3(b) are analyzed to obtain nearly the same results for various combinations of $\{m, m'\}$ as in the analyses of P_∞ . For simplicity the analysis of S for $\{m = 20, m' = 9\}$ is shown in figure 3(c). The $n_c (= 0.7223(1))$ obtained is nearly the same as that obtained from the data in figure 2(c). We also obtain $\gamma_2 = \gamma_2^- = \gamma_2^+ = 0.88(1)$, in which no asymmetry is found between the disordered phase ($n_c < n$) and the ordered phase $n_c > n$. The result $\gamma_2 = 0.88(1)$ is also consistent with the previous estimate $\gamma_2 = 0.88(10)$ on the ER graph. We also obtain $\gamma_2 = 0.88(3)$ from the simulation on the Bethe lattice with $z = 6$. These results clearly show that the obtained values of β_2 and γ_2 for AP_2 are significantly different from those for the random percolation [1].

4. AP_3 model

The triangular cactus was first introduced by Fisher and Essam [17] to investigate the effect of loops [14] on percolation. As shown in figure 4(a), each site in the Bethe lattice is replaced with a triangle of three sites to form the triangular cactus. Thus, the dimensionality of the triangular cactus is infinite as the Bethe lattice. Moreover, the triangular cactus is exactly tripartite, not bipartite, as shown in figure 4(b). Therefore, it is expected that AP_2 on the triangular cactus

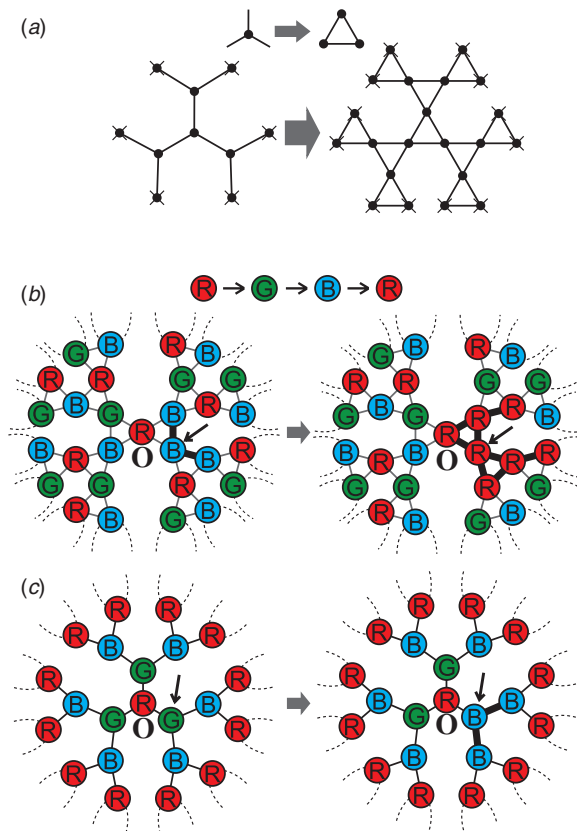


Figure 4. (a) Formation of the triangular cactus from the Bethe lattice with $z = 3$. In the cactus, each site of the Bethe lattice is replaced with a triangle of three sites. Each edge site which is denoted by ‘X’ is connected to an infinite branch. (b) AP₃ on the triangular cactus. The triangular cactus is tripartite as shown in the figure. If a blue(B)-colored cluster indicated by an arrow is selected as in the left figure, it agglomerates all R neighbors and becomes an R cluster by the rule, $R \rightarrow G \rightarrow B \rightarrow R$ as in the right figure. (c) AP₃ on the Bethe lattice. The cluster growth method in AP₃ on the Bethe lattice is the same as that explained in (b).

belongs to the universality class of the random percolation as AP₂ on the two-dimensional triangular lattice [10].

Recently, Lau *et al* suggested the AP_k model, which breaks the Z_k symmetry of the k -partite graph [13]. It was conjectured that the universality class of AP_k depends on k [13]. However, AP_k for $k \geq 3$ has never been quantitatively studied, yet. In this section, AP₃ on the triangular cactus is studied to obtain the MFT of AP₃. The Bethe lattice can also be considered tripartite, four-partite and so on, as well as bipartite. Thus, AP_k on the Bethe lattice belongs to a unique universality class depending on k , because AP_k model breaks the Z_k symmetry of the Bethe lattice without breaking other symmetries. Thus, one can confirm the validity of the MFT of AP₃ by comparing the critical exponents of AP₃ on the Bethe lattice to those of AP₃ on the triangular cactus.

To make the AP₃ model which breaks the Z_3 symmetry of a tripartite graph such as the triangular cactus, the cluster growth process in the AP₃ model should be carefully defined [13]. In the growth process, one cluster is randomly selected and the selected cluster merges some of the nearest-neighboring clusters into a new cluster, instead of all neighboring clusters.

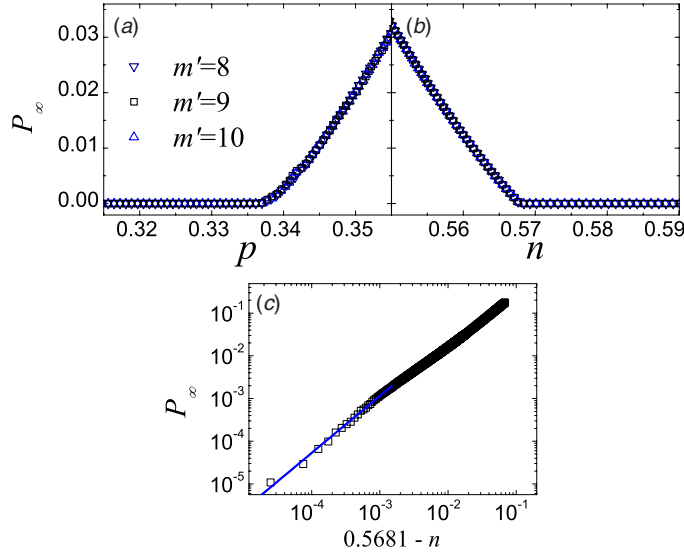


Figure 5. Order parameter P_∞ for AP_3 from the self-consistent simulation on the infinite triangular cactus using the m -generation cactus and m' -generation cactus. (a) Plots of P_∞ against p for the combinations $\{m = 20, m' = 8\}$ (∇), $\{m = 20, m' = 9\}$ (\square) and $\{m = 20, m' = 10\}$ (\triangle). (b) Plots of P_∞ against n for the same combinations in (a). (c) Plots of P_∞ against $n_c - n$ with $n_c = 0.5681$ for the combination $\{m = 20, m' = 9\}$. The line denotes the relation $P_\infty \simeq (n_c - n)^{\beta_3}$, with $n_c = 0.5681$ and $\beta_3 = 1.35$.

In a tripartite graph, initially, three colors are arranged such that no pair of nearest-neighbor sites has the same color. Therefore, we can identify a cluster by colors such as red (R), green (G) and blue (B). In AP_3 , a selected B cluster is defined to agglomerate only the neighboring R clusters to become a larger R cluster as shown in figure 4(b). A G-cluster merges only the neighboring B clusters and an R cluster merges only G clusters based on a cyclic rule, $R \rightarrow G \rightarrow B \rightarrow R$ [13]. The rule $R \rightarrow B \rightarrow G \rightarrow R$ can also be used, but it cannot be physically different from the model with the $R \rightarrow G \rightarrow B \rightarrow R$ rule. The cluster-merging process of AP_3 on the Bethe lattice is the same as that on the triangular cactus as shown in figure 4(c).

For the MFT of AP_3 the self-consistent simulation for AP_3 on the infinite triangular cactus is carried out. The self-consistent simulation is almost the same as that for AP_2 on the Bethe lattice. First consider the m -generation triangular cactus from \mathbf{O} , which has $N_m = 2^{m+2} - 3$ sites. To make a complete infinite triangular cactus, one should add an infinite branch to each of 2^{m+1} edge sites. Other details of the self-consistent simulation on the infinite triangular cactus are exactly the same as those on the Bethe lattice.

We first explain the results of the self-consistent simulation for AP_2 on the infinite triangular cactus using the combination of $\{m = 20, m' = 9\}$. From the data for P_∞ and S , n_c , β and γ are estimated as $n_c = 0.6761(1)$, $\beta = 1.01(2)$ and $\gamma = 1.00(2)$. This result supports the expectation in [13] that AP_2 on a tripartite graph belongs to the random percolation universality class with $\beta = \gamma = 1$.

In contrast, AP_3 on a tripartite graph should belong to a new universality class. The results of the self-consistent simulation for AP_3 on the infinite triangular cactus using the m -generation cactus and m' -generation cactus are displayed in figures 5 and 6. The combinations of $\{m, m'\}$ used for the simulations are also $\{m = 20, m' = 8\}$, $\{m = 20, m' = 9\}$ and $\{m = 20, m' = 10\}$.

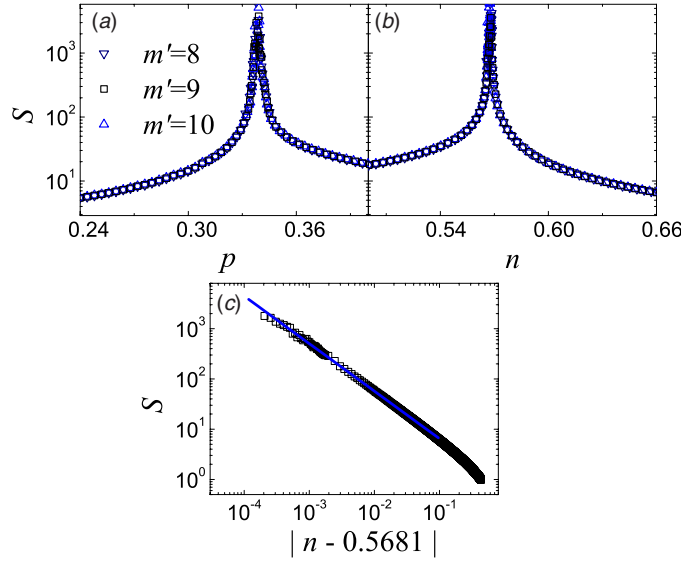


Figure 6. Average size S of finite clusters for AP_2 from the self-consistent simulation on the triangular cactus using the m -generation cactus and m' -generation cactus. (a) Plots of S against p for the combinations $\{m = 20, m' = 8\}$ (∇), $\{m = 20, m' = 9\}$ (\square) and $\{m = 20, m' = 10\}$ (\triangle). (b) Plots of S against n for the same combinations as in (a). (c) Plots of S against $|n_c - n|$ with $n_c = 0.5681$ for the combination $\{m = 20, m' = 9\}$. The line denotes the relation $S \simeq |n - n_c|^{-\gamma_3}$ for $n > n_c$ with $n_c = 0.5681$ and $\gamma_3 = 0.94$.

In figures 5(a) and 6(a), the results of the self-consistent simulation are also displayed for p as in figures 2(a) and 3(a). But the main analyses are done by the use of n as in section 3. The simulation results for AP_3 on the triangular cactus are also nearly identical for various m' ($= 8, 9, 10$) (see figures 5(a), (b) and 6(a) and 6(b)) as in section 3.

By using the same equations as (10) and (11), the data for the combination of $\{m = 20, m' = 9\}$ in figures 5(c) and 6(c) are analyzed. From the analysis, the order parameter exponent β_3 and the susceptibility exponent γ_3 of AP_3 on the triangular cactus are obtained as $\beta_3 = 1.35(5)$ and $\gamma_3 = 0.94(2)$ with $n_c = 0.5681(1)$. We also obtain nearly the same β_3 , γ_3 and n_c within the estimated error by analyzing the data for $m' = 10$ and $m' = 8$ in figures 5(b) and 6(b).

The results of the simulations for AP_3 on the Bethe lattice with $z = 3$ using the same combinations $\{m = 20, m' = 8\}$, $\{m = 20, m' = 9\}$ and $\{m = 20, m' = 10\}$ are displayed in figures 7 and 8. The obtained exponents of AP_3 on the Bethe lattice are the same as those of AP_3 on the triangular cactus as shown in figures 7(c) and 8(c). β_3 and γ_3 obtained from the simulations on the triangular cactus and on the Bethe lattice are the first numerical results for AP_3 .

β_3 and γ_3 satisfy the inequalities $\beta_\infty < \beta_3 < \beta_2$ and $\gamma_\infty > \gamma_3 > \gamma_2$, where $\beta_\infty (=1)$ and $\gamma_\infty (=1)$ are the MFT exponents of the random percolation. The inequalities suggest that the MFT exponents of AP_k approach to those of the random percolation as k increases.

5. Summary

Finding the exact MFT is the first step to understand the various physical properties of a new model. AP was suggested as a natural extension of the random percolation model. Some

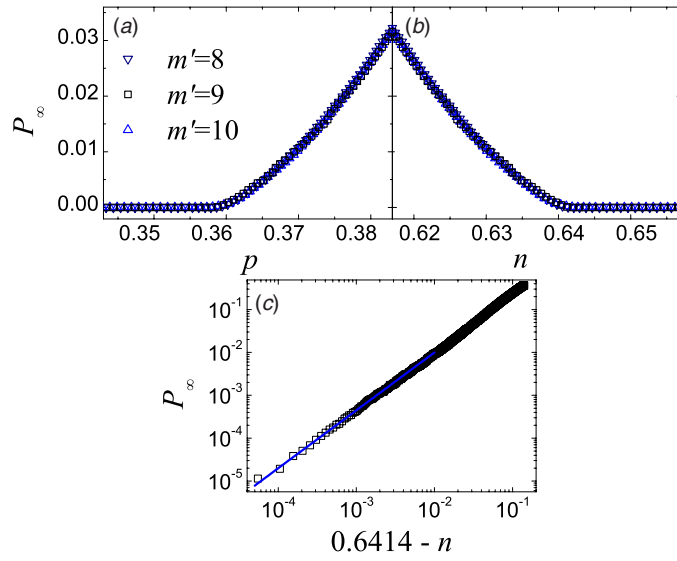


Figure 7. Order parameter P_∞ for AP_3 from the self-consistent simulation on the Bethe lattice with $z = 3$ using the m -generation tree and m' -generation tree. (a) Plots of P_∞ against p for the combinations $\{m = 20, m' = 8\}$ (∇), $\{m = 20, m' = 9\}$ (\square) and $\{m = 20, m' = 10\}$ (\triangle). (b) Plots of P_∞ against n for the same combinations as in (a). (c) Plots of P_∞ against $n_c - n$ with $n_c = 0.6414$ for the combination $\{m = 20, m' = 9\}$. The line denotes the relation $P_\infty \simeq (n_c - n)^{\beta_3}$, with $n_c = 0.6414$ and $\beta_3 = 1.35$.

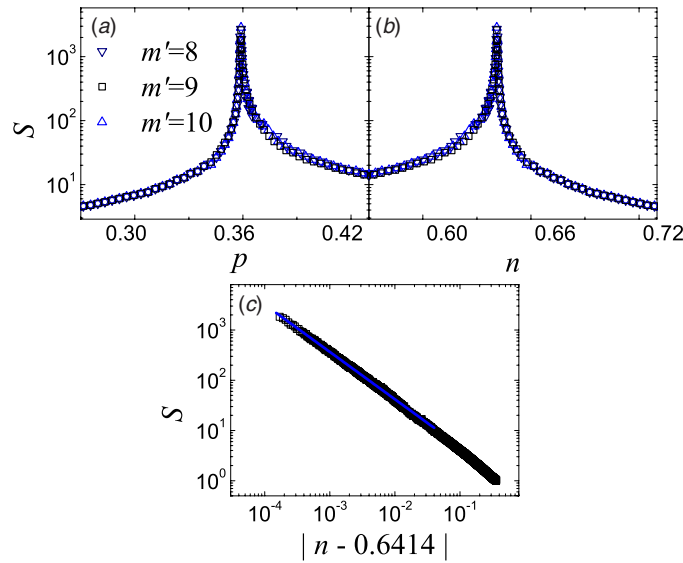


Figure 8. Average size S of finite clusters for AP_2 from the self-consistent simulation on the Bethe lattice with $z = 3$ using the m -generation tree and m' -generation tree. (a) Plots of S against p for the combinations $\{m = 20, m' = 8\}$ (∇), $\{m = 20, m' = 9\}$ (\square) and $\{m = 20, m' = 10\}$ (\triangle). (b) Plots of S against n for the same combinations in (a). (c) Plots of S against $|n_c - n|$ with $n_c = 0.6414$ for the combination $\{m = 20, m' = 9\}$. The line denotes the relation $S \simeq |n - n_c|^{-\gamma_3}$ for $n > n_c$ with $n_c = 0.6414$ and $\gamma_3 = 0.94$.

Table 1. Critical thresholds and critical exponents for the AP₂ and AP₃ models on the Bethe lattice and the infinite triangular cactus.

Model	Graph	n_c (or p_c)	β	γ
AP ₂	Bethe lattice ($z = 3$)	$n_c = 0.7221(1)$	1.79(3)	0.88(1)
AP ₂	Triangular cactus	$n_c = 0.6761(1)$	1.01(1)	1.00(2)
AP ₃	Bethe lattice ($z = 3$)	$n_c = 0.6414(1)$	1.35(5)	0.94(2)
AP ₃	Triangular cactus	$n_c = 0.5681(1)$	1.35(5)	0.94(2)
Random percolation	Bethe lattice ($z = 3$)	$p_c = 1/2$	1	1

numerical studies for AP₂ were performed on lower-dimensional lattices and random graphs [9–13]. Based on those numerical studies, it was conjectured that AP_k belongs to a new universality class depending on k if the cluster growth breaks the Z_k symmetry of a k -partite graph. However, the mean-field approach based on the evolutionary dynamics of clusters did not agree with the numerical simulations. This strongly indicates that AP is not fully understood even at the mean-field level [12].

Therefore, in order to provide the exact MFT, we apply the self-consistent simulation method [8] to AP₂ on the Bethe lattice and AP₃ on the triangular cactus and the Bethe lattice. From the direct and precise measurement of P_∞ and S through the self-consistent simulation, we obtain $\beta_2 = 1.79(3)$ and $\gamma_2 = 0.88(1)$ on the Bethe lattice when the Z_2 symmetry is broken spontaneously at the transition threshold. Similarly, we obtain $\beta_3 = 1.35(5)$ and $\gamma_3 = 0.94(2)$ both on the triangular cactus and on the Bethe lattice if the Z_3 symmetry is broken spontaneously. In contrast, the AP₂ model on the triangular cactus gives $\beta = 1.01(2)$ and $\gamma = 1.00(2)$. This result shows that AP₂ on a tripartite graph belongs to the same universality class as that of the random percolation. These results are summarized in table 1.

Therefore, the results for AP₃ on the triangular cactus and the Bethe lattice provide the exact MFT, which verifies the Lau *et al* conjecture [13] quantitatively. In addition, by comparing the obtained critical exponents with those of the random percolation, we also find the inequalities $\beta_\infty < \beta_3 < \beta_2$ and $\gamma_\infty > \gamma_3 > \gamma_2$. These inequalities also quantitatively verify the conjecture that the universality class of AP_k depends on k [13].

Acknowledgments

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