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Fluctuation of incoming flux with multiplicative noise on a scale-free network

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Abstract. – We study the influence of topology on the dynamical properties of a diffusion process which can be described by a diffusion equation with multiplicative noise on a complex network. From numerical simulations we find that the fluctuation of the incoming mass on a given node of network scales with the average incoming mass, or flux, in a topology-dependent fashion. By combining numerical results with the Langevin equation of the associated process, we show that inhomogeneity of the underlying structure leads to the appearance of distinct dynamical regions in the system and a crossover behavior in the scaling of fluctuations with average flux.

Nonequilibrium statistical mechanics has been an intensively studied subject [1,2], in various physical systems [3–10]. One example of nonequilibrium phenomena which has received great attention in the last decade is self-organized criticality with potential relevance to the scale-invariant features observed in many systems such as the sand pile model [11], river networks [12–14] and growth dynamics [5]. An early version of self-organized criticality was proposed by Takayasu [15,16]. The model is believed to be relevant to economic systems because its diffusion process with aggregation, deposition and evaporation of particles resembles the "efficiency" dynamics of competing agents in economic systems [17]. In many cases such as population dynamics or wealth dynamics, it is more natural to assume that the amount of diffusion can be described by the multiplicative noise [18]. Moreover, it is well known that the diffusive systems with multiplicative noise can be characterized by generalized Levy-Pareto distribution [18–21] and most of the previous works have been focused on the generation of power law by multiplicative noise.

In order to make a direct comparison between a model system and empirical observations, however, one should understand how the underlying interaction topology between agents affects their dynamical behavior. The effects of the underlying topology of dynamical properties such as relaxation time, the autocorrelation function and the return probability of random

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walkers have been recently studied [22,23]. It is especially known that diffusive particles constrained to move along the nodes and links of a scale-free network tend to agglomerate on the high connectivity nodes or hubs [24]. Moreover, it has been found that the average number of incoming walkers, $\langle H_i \rangle$, reaching a node i on a reasonably large time window $\tau \ll T_{max}$, scales with the fluctuations around the average σ as $\sigma \sim \langle H_i \rangle^{\alpha}$, where $\alpha = 1/2$ if the total number of walkers in the system does not fluctuate too much between different time windows, and $\alpha = 1$ otherwise [25] ($\langle \cdots \rangle$ represents average over a time window τ and T_{max} is the observation time). Thus, the amplitude ΔW of the variations of the average number of walkers in the system determines the value of the exponent α. It is important to understand the relation between averages and the width of fluctuations, as the relative fluctuations of the activity on a node can vary widely depending on such scaling. Since real-world systems often have limited node capacity (such as traffic on highways, byte packets on the Internet router and other), jamming and malfunctioning can result from unbounded fluctuations. Here we show that, if the diffusive process is multiplicative and the underlying topology is intrinsically inhomogeneous, there is a crossover from $\alpha = 1/2$ to $\alpha = 1$ on the $\langle H \rangle$ *vs.* σ curve even for $\Delta W = 0$. While this crossover is evident for multiplicative diffusive processes on a scale-free network, it does not appear on random Erdös-Rényi (ER) networks for which the connectivity distribution is narrowly peaked at $\langle k \rangle$.

In the Takayasu model, a non-negative mass variable h_i is assigned to each site i of a lattice. At each time step the total mass of a randomly selected site l moves to one of its nearest neighbors j and aggregates with rate 1 resulting $h'_j = h_j + h_l$. In addition, a unit mass is deposited at a randomly chosen site with rate q [15,16]. Majundar *et al.* [17,26] generalized the model by including desorption of a unit mass from a randomly chosen site with rate p and showed that in the steady state the mass distribution $P(h)$ follows a power law for $q < q_c(p)$ and decays exponentially for $q > q_c(p)$, with [17,26]

$$q_c(p) = p + 2 - 2\sqrt{p+1}. \tag{1}$$

In our study we assume that the amount of mass moving from a node i to one of its nearest neighbors is a random fraction of the total mass at node i. Thus, we adopt the following set of rules for the diffusive process: starting from a random initial distribution of mass h_i at each node i, i) a unit of mass is deposited at a randomly chosen site i with probability $p/(p+q+1)$, ii) a unit mass is removed from a randomly chosen site i with probability $q/(p+q+1)$, and iii) a random fraction of mass from a randomly chosen site i moves to one of its (randomly chosen) nearest neighbors j with probability $1/(p+q+1)$.

We also define the flux $h(t)$ at node i on time t as

$$H_i(t) = \sum_{t'=0}^{\tau} h_i^{in}(t+t'), \tag{2}$$

where $h_i^{in}(t)$ is the incoming mass at time t and τ is the size of the time window representing the level of coarse graining of the measurement time. Thus, we obtain for each node i a time series $\{H_i(0), H_i(\tau), \cdots, H_i(T_{max})\}$, where T_{max} is the total observation time and $T_{max} \gg \tau$. We simulated this process on a Barabási-Albert (BA) scale-free network [27] with $N = 10^5$ nodes, setting the deposition and evaporation rates q and p to satisfy the conditions: i) $\sum_i^N H_i(t) \to$ const (steady-state condition) and ii) $q \simeq q_c$, with $(p+q)/(p+q+1) = 0.1$ (eq. (1)). Moreover, as long as there is a diffusion, the scaling behavior is governed by it [28]. Only in the limit that the diffusion rate goes to zero, the additive noise dominates the dynamics, and $\alpha \to 1$. The total observation time T_{max} and the measurement time window τ are set to 50000 and

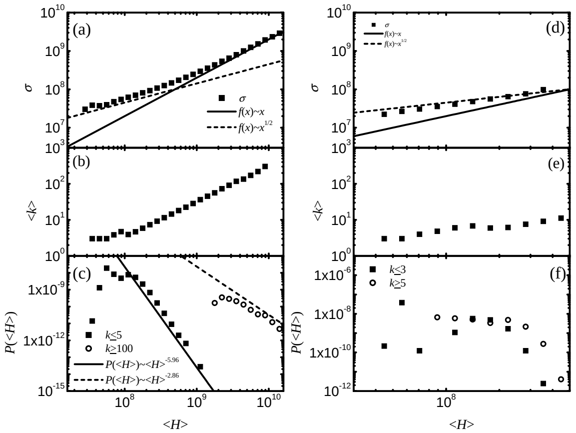

Fig. 1 – Plots for the diffusion-dominant region on SF networks((a)-(c)) and on ER networks((d)-(f)). (a) and (d) show the measurement of fluctuations of incoming fluxes on a SF network and a random network, respectively. Solid lines denote $\sigma \sim \langle H \rangle$ and dotted lines represent $\sigma \sim \langle H \rangle^{1/2}$. (b) and (e) average connectivity $\langle k \rangle$ as a function of average incoming flux $\langle H \rangle$. Note that in the ER graph, the largest value of $\langle k \rangle$ is around 10, corresponding to the $\alpha = 1/2$ region in SF networks. (c) and (f) average incoming flux distribution. Solid squares (■) represent low-connectivity regions and open circles (○) denote high-connectivity regions.

100 Monte Carlo time steps, respectively. Averaging $h_i(t)$ over time and calculating σ_i for each node i, we find that $\sigma \sim \langle H \rangle^{\alpha}$, where $\alpha = 1/2$ for small $\langle H \rangle$ values, and $\alpha = 1$ for large $\langle H \rangle$ (fig. 1(a)). As shown in ref. [24], high connectivity nodes are subject to more traffic than low connectivity nodes are, the traffic fluctuations being amplified by the multiplicative nature of the process, that is, the bigger the mass is, the larger the amount of mass transferred to the nodes is, on the average. We check this by measuring the average connectivity $\langle k \rangle$ of a node receiving an amount of mass $\langle H \rangle$, shown in fig. 1(b), confirming that hubs receive more traffic. From figs. 1(a) and (b) one can find $\langle k \rangle_c$, the value of k associated with the value of $\langle H \rangle$ where the crossover occurs, and calculate the restricted distributions of mass $P(\langle H \rangle)_{k<k_c}$ and $P(\langle H \rangle)_{k>k_c}$. As depicted in fig. 1(c), these distributions differ fundamentally, the low-connectivity ($k < k_c$) distribution being much narrower than the high-connectivity one ($k > k_c$). It is this broad range in $P(\langle H \rangle)_{k>k_c}$ that makes the average mass $\langle H \rangle$ scales linearly with the mass fluctuations when $\langle H \rangle \gg \langle H \rangle_c$ (ref. [25]). We verified this by repeating the same experiment on a Erdös-Rényi (ER) random network. In this case the crossover disappears (fig. 1(d)), the average mass scales weakly with connectivity (fig. 1(e)), and the restricted mass distributions for low and high connectivity are both narrowly distributed (fig. 1(f)), which is reflected in the $\alpha = 1/2$ exponent measured over the whole range of $\langle H \rangle$ (fig. 1(f)).

From these numerical simulations we conclude that the topological differences between scale-free and random networks lead to different dynamical behaviors. This can be accredited to the multiplicative nature of the diffusive process, since simple diffusion on scale-free networks leads to $\alpha = 1/2$, as reported in ref. [25].

To gain a deeper understanding of the crossover observed on scale-free networks, we solved the Langevin equation of the dynamical process by a mean-field approximation. Since, by definition, the incoming flux is a random fraction of the nearest-neighbors' mass, the incoming flux is on average proportional to the mass at the nearest neighbors. The Langevin equation for the change of mass at node i during a unit time interval is

$$H_i\,(t+1) = H_i\,(t) \;+\; \sum_j^{k_i} \frac{1}{k_j}\eta_j\,(t)\,H_j\,(t) - \eta_i\,(t)\,H_i\,(t) + \xi_j(t), \tag{3}$$

where the second term on the right-hand side represents the incoming mass from the nearest neighbors and the third term the outgoing mass from node i during the unit time interval. The last term represents fluctuations caused by deposition and desorption. Moreover, $\eta_i(t)$ is a random variable between the interval [0,1] and $\xi_i(t)$ is assumed to be Gaussian white noise uncorrelated both in space and time. This type of diffusion equation with multiplicative noise has been studied in the context of stochastic Lotka-Volterra models, which can be characterized by the (truncated) Pareto or Lévy distribution $P(H) \sim H^{-1-\mu}$ [18–20]. Since we are focusing only on the incoming mass in the diffusion dominant region, eq. (3) can be reduced to

$$H_i(t+1) = H_i(t) + \sum_j^{k_i} \frac{1}{k_j}\eta_j\,(t)\,H_j\,(t)\,. \tag{4}$$

In the continuum limit we can approximate the change of incoming mass by

$$\frac{\mathrm{d}H}{\mathrm{d}t} \simeq \sum_j^{k_i} \frac{1}{k_j}\eta_j\,(t)\,H_j\,(t)\,. \tag{5}$$

The data in fig. 2 shows that the incoming flux distribution $P(H)$ also has a long power law decaying tail even when the outgoing flux is not considered. The value of the exponent μ is known to depend on the details of the model's parameters [18–20]. By least-square fitting, we find the corresponding exponent to be $-1-\mu \simeq -3.3$ in our model.

In order to obtain an approximate expression for $H(t)$ we assume that $\sum_j^{k_i} \frac{1}{k_j}\eta_j(t)H_j(t) \simeq \langle k_i\rangle \frac{1}{\langle k_{nn}\rangle}\langle \eta_j(t)H_j(t)\rangle$, where $\langle k_{nn}\rangle$ denotes the average degree of a node's nearest neighbors. Since $\eta_j(t)$ and $H_j(t)$ are independent variables, eq. (5) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t} \simeq \frac{\langle k\rangle}{\langle k_{nn}\rangle}\langle\eta\rangle\langle H_{nn}\rangle \equiv A(\langle k\rangle)\langle H_{nn}\rangle\,, \tag{6}$$

where $\langle H_{nn}\rangle$ represents the average incoming mass on the nearest neighbors of the node. In general, the average connectivity of the nearest neighbors of a given node can be expressed as a function of connectivity of the chosen node, *i.e.* $\langle k_{nn}\rangle \sim \langle k\rangle^{-\nu}$. For the BA model ν becomes 0 [29]. To express $\langle H_{nn}\rangle$ as a function of the incoming mass of a selected node, we calculated $\langle H_{nn}\rangle$ numerically, as depicted in fig. 3.

As shown in fig. 3, $\langle H_{nn}\rangle$ decreases rapidly as $\langle H\rangle$ increases and then converges slowly to a constant value. The crossover between the rapidly decaying region and the almost flat region occurs around $H_c \simeq 10^8$, and this value is consistent with the value obtained from fig. 1(a). Based on the data in fig. 3, we approximated $\langle H_{nn}\rangle = \text{const}$ for $H \geq 10^8$ and

$$\langle H_{nn}\rangle \approx \left(a + \frac{b}{c+H}\right), \tag{7}$$

for $H \leq 10^8$. Here a, b and c are the fitting parameters. Using least-square fitting we obtained the value of the parameters a, b, and c as $a = 4.52581\times10^8$, $b = 5.7908\times10^{15}$, and

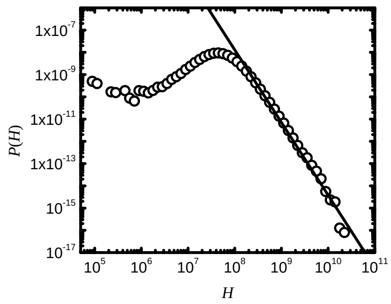

Fig. 2 – Plot of incoming flux distribution $P(H)$. The slope of the solid line represents $-1-\mu \simeq -3.3$.

$c = 2.58341 \times 10^6$. Equation (7) represents the fact that the large flux going into the nearest neighbors of the selected node can decrease the flux coming into the selected node.

The calculation for $H \geq 10^8$ with the approximation $\langle H_{nn} \rangle = \text{const}$ gives

$$\langle H \rangle \sim \langle k \rangle \,, \tag{8}$$

and

$$\sigma \sim \langle H \rangle \,. \tag{9}$$

Equation (8) indicates that the linear dependence of $\langle k \rangle$ on $\langle H \rangle$ for large $\langle H \rangle$ values in fig. 1(b) comes from the nearly constant value of $\langle H_{nn} \rangle$, leading to the linear dependence of σ on $\langle H \rangle$, or $\alpha = 1$.

On the other hand, for $\langle H \rangle \leq \langle H \rangle_c$, eq. (6) combined with the approximation (7) and the conditions $a/b \ll 1$, $\exp[-(a^2 A/b)t] \ll 1$ in the large-t limit and $\langle H \rangle \geq 0$, we obtain

$$H(\langle k \rangle \,, t) \;=\; D_0 + D_1 e^{-qt}, \tag{10}$$

and

$$\langle H \rangle = \frac{1}{T} \sum_{t}^{T} H(\langle k \rangle \,, t) \simeq D_0 - \frac{D_1}{Tq} \left(e^{-qT} - 1 \right) . \tag{11}$$

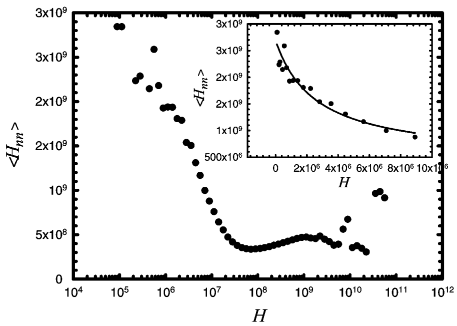

Fig. 3 – Plot of $\langle H_{nn} \rangle$ against H. The inset shows $\langle H_{nn} \rangle$ for $k \leq 10^8$ The solid line represents the hyperbolic decaying function $\langle H_{nn} \rangle = a + b/(c + H)$.

Here $D_0 = c\left(\frac{ac}{\sqrt{b(ac+b)}} - 1\right)$, $D_1 = \frac{b(ac+b)}{a^2c}$ and $q = a^2A/b$. Therefore, the fluctuation in the incoming mass is given by

$$\sigma \simeq \sqrt{\frac{D_1^2}{2qT}\,(1 - e^{-2qT}) + \frac{D_1^2}{T^2q^2}\,(1 - e^{-qT})}. \tag{12}$$

In the limit $T \to \infty$, the dominant term in eq. (12) scales with $(1/T)^{1/2}$. From eq. (11), we know that the average incoming flux $\langle H \rangle$ is proportional to $(1/T)$, and we conclude that $\sigma \sim \langle H \rangle^{1/2}$, as expected.

In summary, we have studied a diffusion process reminiscent of Takayasu's model on scale-free and Erdös-Rényi networks. On this process, the number of particles on a given node dictates how many particles will diffuse simultaneously from it. This type of process can be described by a diffusion equation with multiplicative noise, generally characterized by Pareto or Lévy distributions and is regarded as a model for systems such as population dynamics of each species in a given region or stock exchange markets in the financial systems. Through numerical simulations of the model, we found a nontrivial crossover between two different scalings of the average number of incoming particles on a node $\langle H \rangle$ and fluctuations about the average σ. We conclude that they occur due to the multiplicative nature of the process coupled with the intrinsic inhomogeneity of the underlying scale-free topology. This result is also verified by solving the Langevin equation of the process. Comparisons with a simpler, random geometry show how topology affects dynamics, since this crossover is not present on the latter.

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