Ordinary Differential Equations

Soon-Hyung Yook

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Euler Method

The most easiest and intuitive method.
Solve the equation:

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

From the Taylor expansion of y around x

$$y(x + h) = y(x) + hy' + \frac{1}{2}h^2y'' + \cdots$$

Since $y' = f(x, y)$,

$$y(x + h) = y(x) + hf(x, y) + \mathcal{O}(h^2)$$

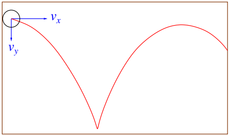
or

Euler Method

$$y_{i+1} = y_i + hf_i + \mathcal{O}(h^2) \tag{2}$$

where $f_i \equiv f(x_i, y_i)$.

Example-Bouncing Ball



Analytic results are known:

$$v_x = v_{x_0} \qquad x = v_0 t + x_0$$

$$v_y = v_{y_0} - gt \qquad y = y_0 + v_{y_0} t - \frac{1}{2} g t^2$$

Differential Equations

Solve two sets of differential equations:

$$\frac{dv_x}{dt} = 0 \qquad \frac{dx}{dt} = v_x$$

$$\frac{dv_y}{dt} = -g \qquad \frac{dy}{dt} = v_y$$

Example-Bouncing Ball

- discretize with time interval Δt

$$(x_0, y_0), (x_1, y_1), \cdots, (x_i, y_i), \cdots$$

$$(v_{x_0}, v_{y_0}), (v_{x_1}, v_{y_1}), \cdots, (v_{x_i}, v_{y_i}), \cdots$$

- Using Euler method:

$$\left(\begin{array}{l} v_{x_i} = v_{x_{i-1}} (= v_{x_0}) \\ v_{y_i} = v_{y_{i-1}} - g\Delta t \end{array} \right.$$

and

$$\left(\begin{array}{l} x_i = x_{i-1} + v_{x_i} \Delta t \\ y_i = y_{i-1} + v_{y_i} \Delta t \end{array} \right.$$

Homework

Damped Harmonic Montion: Solve the second order differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

using the Euler method. There are three different types of damping:

1. $c^2 - 4mk > 0$ overdamping
2. $c^2 - 4mk = 0$ critical damping
3. $c^2 - 4mk < 0$ underdamping

Plot x vs. t for each case.

Picard Method

Eq. (1) can be expressed as

$$y_{i+1} = y_i + \int_{x_i}^{x_i+h} f(x, y)dx. \quad (3)$$

Use the trapezoidal method in "Numerical Calculus" for the integral in Eq. (3):

Picard Method

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) + \mathcal{O}(h^3) \quad (4)$$

Homework

Damped Harmonic Montion: Solve the second order differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

using the Picard method. There are three different types of damping:

1. $c^2 - 4mk > 0$ overdamping
2. $c^2 - 4mk = 0$ critical damping
3. $c^2 - 4mk < 0$ underdamping

Plot x vs. t for each case.

Predictor-Corrector Method

1. Apply a less accurate algorithm to predict the next value y_{i+1} (Predictor)
   for example, Euler method of Eq. (2)
2. Apply a better algorithm to improve the new value (Corrector)
   for example, Picard method of Eq. (4)

Predictor-Corrector Method

1. calculate the predictor

$$y_{i+1} = y_i + hf_i \equiv p(y_{i+1})$$

2. apply the correction by Picard method

$$y_{i+1} = y_i + \frac{h}{2}\left(f_i + f_{i+1}(x_{i+1}, p(y_{i+1}))\right).$$

Homework

Damped Harmonic Montion: Solve the second order differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

using the Predictor-Corrector method. There are three different types of damping:

1. $c^2 - 4mk > 0$ overdamping
2. $c^2 - 4mk = 0$ critical damping
3. $c^2 - 4mk < 0$ underdamping

Plot x vs. t for each case.

Runge-Kutta Method

To solve the differential equation:

$$\frac{dy}{dt} = f(y,t)$$

we expand $y(t+\tau)$ in terms of the quantities at t with the Taylor expansion:

$$y(t+\tau) = y + \tau y' + \frac{\tau^2}{2}y'' + \frac{\tau^3}{3!}y^{(3)} + \frac{\tau^4}{4!}y^{(4)} + \cdots \tag{5}$$

Let

$$f_{yt} \equiv \frac{\partial^2 f}{\partial y \partial t}$$

and so on. Then

$$y' = f(y,t) \tag{6}$$

$$y'' = f_t + f f_y \tag{7}$$

Runge-Kutta Method

Higher order terms

$$y^{(3)} = f_{tt} + 2ff_{ty} + f^2 f_{yy} + ff_y^2 + f_t f_y \tag{8}$$

and

$$\begin{aligned} y^{(4)} &= f_{ttt} + 3ff_{tty} + 3f_t f_{ty} + 5ff_y f_{ty} + (2+f)ff_{tyy} + 3ff_t f_{yy} \\ &\quad +4f^2 f_y f_{yy} + f^3 f_{yyy} + f_t f_y^2 + ff_y^3 + f_{tt} f_y \end{aligned} \tag{9}$$

Runge-Kutta Method

$y(t+\tau)$ also can be written as

$$y(t+\tau) = y(t) + \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_m c_m \tag{10}$$

with

$$\begin{aligned} c_1 &= \tau f(y,t), \\ c_2 &= \tau f(y + \nu_{21} c_1, t + \nu_{21}\tau), \\ c_3 &= \tau f(y + \nu_{31} c_1 + \nu_{32} c_2, t + \nu_{31}\tau + \nu_{32}\tau) \\ &\vdots \end{aligned} \tag{11}$$

$$c_m = \tau f\left(y + \sum_{i=1}^{m-1} \nu_{mi} c_i, t + \tau \sum_{i=1}^{m-1} \nu_{mi}\right). \tag{12}$$

where α_i $(i = 1, 2, \cdots, m)$ and ν_{ij} $(i = 2, 3, \cdots, m$ and $j < i)$ are parameters to be determined.

Second-Order Runge-Kutta Method

If only the terms up to $\mathcal{O}(\tau^2)$ are kept in Eq. (5),

$$y(t+\tau) = y + \tau f + \frac{\tau^2}{2}(f_t + f f_y). \tag{13}$$

Truncate Eq. (10) up to the same order at $m = 2$:

$$y(t+\tau) = y(t) + \alpha_1 c_1 + \alpha_2 c_2 \tag{14}$$

From Eq. (11),

$$\begin{aligned} c_1 &= \tau f(y, t), \\ c_2 &= \tau f(y + \nu_{21} c_1, t + \nu_{21}\tau). \end{aligned} \tag{15}$$

Second-Order Runge-Kutta Method

Now expand c_2 up to $\mathcal{O}(\tau^2)$:

$$c_2 = \tau f + \nu_{21}\tau^2(f_t + f f_y) \tag{16}$$

From Eqs. (14)-(16) we obtain

$$y(t+\tau) = y(t) + (\alpha_1 + \alpha_2)\tau f + \alpha_2\tau^2\nu_{21}(f_t + f f_y). \tag{17}$$

By comparing Eq. (17) with Eq. (13), we have

$$\alpha_1 + \alpha_2 = 1, \tag{18}$$

and

$$\alpha_2\nu_{21} = \frac{1}{2}. \tag{19}$$

Two equations with three unknowns.

Second-Order Runge-Kutta Method

Choose $\alpha_1 = \frac{1}{2}$ then

$$\begin{cases} \alpha_2 = \frac{1}{2} \\ \nu_{21} = 1 \end{cases}$$

2nd-Order Runge-Kutta Method

$$y(t+\tau) = y(t) + \frac{1}{2}\tau f(y,t) + \frac{1}{2}\tau f(y + c_1, t+\tau)$$

with

$$c_1 = \tau f(y,t).$$

2nd-order RK is the same with the *Predictor-Corrector (or Modified Euler Method)*.

Fourth-Order Runge-Kutta Method

If we keep the terms in Eq. (5) and Eq. (10) up to $\mathcal{O}(\tau^4)$ we obtain the *4th-order RK method*.

4th-Order Runge-Kutta Method

$$y(t+\tau) = y(t) + \frac{1}{6}\left(c_1 + 2c_2 + 2c_3 + c_4\right), \tag{20}$$

with

$$\begin{aligned} c_1 &= \tau f(y,t), \\ c_2 &= \tau f\left(y + \frac{c_1}{2}, t + \frac{\tau}{2}\right), \\ c_3 &= \tau f\left(y + \frac{c_2}{2}, t + \frac{\tau}{2}\right), \\ c_4 &= \tau f(y + c_3, t + \tau) \end{aligned} \tag{21}$$

Example

- Ref. Boas 3rd Ed. pp. 402 (4th_RK.c)

Solve the differential equation

$$\frac{dy}{dx} = \frac{2x}{1+x^2} - \frac{6x}{1+x^2}y$$

i.e.

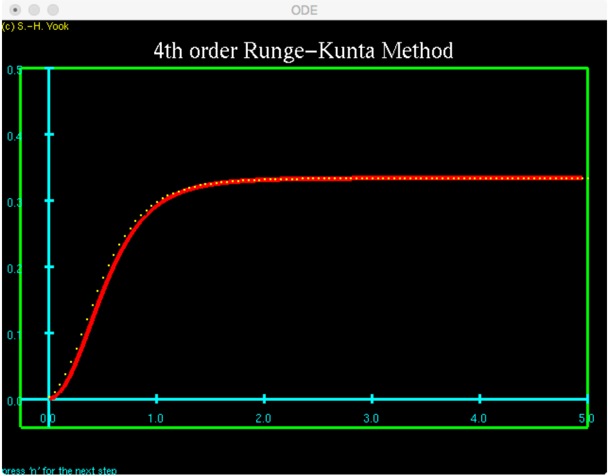
$$f(x,y) = \frac{2x}{1+x^2} - \frac{6x}{1+x^2}y.$$

And compare the numerical result with the known analytic solution,

$$y = \frac{3x^2+3x^4+x^6}{3(1+x^2)^3} + \frac{A}{(1+x^2)^3}.$$

The constant A is determined as $A=0$ when $x=0$ and $y=0$.

Example


ODE
(c) S.-H. Yook
4th order Runge–Kunta Method
0.5
0.4
0.3
0.2
0.1
0.0
0.0
1.0
2.0
3.0
4.0
5.0
press 'n' for the next step

Homework

Solve the differential equation

$$y' + \frac{y}{\sqrt{x^2+1}} = \frac{1}{x+\sqrt{x^2+1}}$$

with initial condition $y = 0$ when $x = 0$.

Solve Newton's Equation of motion

Newton's equation of motion in one-dimensional space

$$\frac{d^2x}{dt^2} = \frac{F(x,v,t)}{m} \equiv f(x,v,t) \tag{22}$$

Disassemble Eq. (22) into two steps:

$$\frac{dv}{dt} = f(x,v,t) \tag{23}$$

and

$$\frac{dx}{dt} = v \tag{24}$$

Solve Newton's Equation of motion

By using the 4th-order Runge-Kutta method

$$v_{i+1} = v_i + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4), \tag{25}$$

where

$$\begin{aligned} c_1 &= \tau f(x_i, v_i, t_i) \\ c_2 &= \tau f\left(x_i + \frac{q_1}{2}, v_i + \frac{c_1}{2}, t_i + \frac{\tau}{2}\right) \\ c_3 &= \tau f\left(x_i + \frac{q_2}{2}, v_i + \frac{c_2}{2}, t_i + \frac{\tau}{2}\right) \\ c_4 &= \tau f(x_i + q_3, v_i + c_3, t + \tau) \end{aligned} \tag{26}$$

Solve Newton's Equation of motion

And from Eq. (24)

$$x_{i+1} = x_i + \frac{1}{6}(q_1 + 2q_2 + 2q_3 + q_4), \tag{27}$$

where

$$\begin{aligned} q_1 &= \tau v_i \\ q_2 &= \tau\left(v_i + \frac{c_1}{2}\right) \\ q_3 &= \tau\left(v_i + \frac{c_2}{2}\right) \\ q_4 &= \tau(v_i + c_3) \end{aligned} \tag{28}$$

From Eq. (27) and Eq. (28)

$$x_{i+1} = x_i + \frac{1}{6}\left[\tau v_i + 2\tau\left(v_i + \frac{c_1}{2}\right) + 2\tau\left(v_i + \frac{c_2}{2}\right) + \tau(v_i + c_3)\right] \tag{29}$$

Or

$$x_{i+1} = x_i + \tau v_i + \frac{\tau}{6}\left[c_1 + c_2 + c_3\right] \tag{30}$$

Solve Newton's Equation of motion

Therefore, we only need to calculate c_i's!

Newton's Equation of Motion

$$x_{i+1} = x_i + \tau v_i + \frac{\tau}{6}\left(c_1 + c_2 + c_3\right)$$

$$v_{i+1} = v_i + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4)$$

where

$$\begin{aligned} c_1 &= \tau f(x_i, v_i, t_i) \\ c_2 &= \tau f\left(x_i + \tfrac{\tau v_i}{2}, v_i + \tfrac{c_1}{2}, t_i + \tfrac{\tau}{2}\right) \\ c_3 &= \tau f\left(x_i + \tfrac{\tau v_i}{2} + \tfrac{\tau c_1}{4}, v_i + \tfrac{c_2}{2}, t_i + \tfrac{\tau}{2}\right) \\ c_4 &= \tau f\left(x_i + \tau v_i + \tfrac{\tau c_2}{2}, v_i + c_3, t + \tau\right) \end{aligned}$$

Example: Van der Pol Oscillator

Van der Pol Oscillator

$$\frac{d^2x}{dt^2} = \mu(x_0^2 - x^2)\frac{dx}{dt} - \omega^2 x$$

or

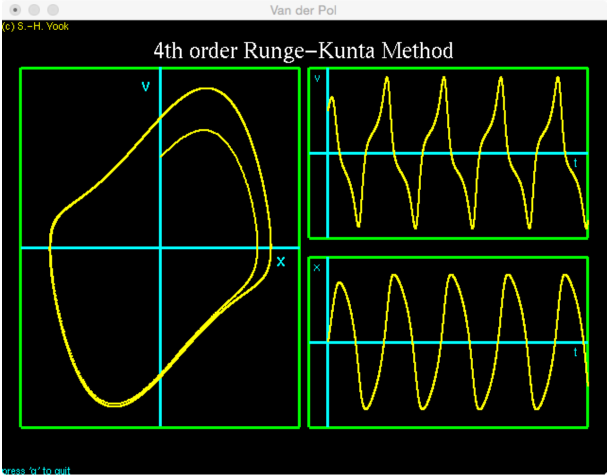
$$\frac{d^2x}{dt^2} = f(t,x,v) = \mu(x_0^2 - x^2)v - \omega^2 x$$

x_0, μ, ω are given constants.

From the previous page:

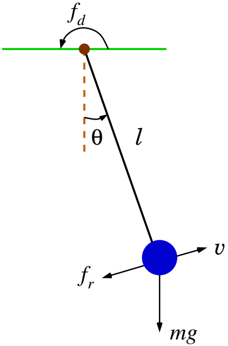
$$\begin{aligned}
x_{i+1} &= x_i + \tau v_i + \tfrac{\tau}{6}\left(c_1 + c_2 + c_3\right) \\
v_{i+1} &= v_i + \tfrac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4) \\
c_1 &= \tau\left[\mu(x_0^2 - x_i^2)v_i - \omega^2 x_i\right] \\
c_2 &= \tau\left[\mu\left(x_0^2 - \left(x_i + \tfrac{\tau v_i}{2}\right)^2\right)\left(v_i + \tfrac{c_1}{2}\right) - \omega^2\left(x_i + \tfrac{\tau v_i}{2}\right)\right] \\
c_3 &= \tau\left[\mu\left(x_0^2 - \left(x_i + \tfrac{\tau v_i}{2} + \tfrac{\tau c_1}{4}\right)^2\right)\left(v_i + \tfrac{c_2}{2}\right) - \omega^2\left(x_i + \tfrac{\tau v_i}{2} + \tfrac{\tau c_1}{4}\right)\right] \\
c_4 &= \tau\left[\mu\left(x_0^2 - \left(x_i + \tau v_i + \tfrac{\tau c_2}{2}\right)^2\right)\left(v_i + c_3\right) - \omega^2\left(x_i + \tau v_i + \tfrac{\tau c_2}{2}\right)\right]
\end{aligned}$$

Example: Van der Pol Oscillator



Homework

Driven Pendulum



A point mass m is attached to the lower end of massless rod of length l. The pendulum is confined to a vertical plane, acted on by a driving force f_d and a resistive force f_r (see the figure). The motion of the pendulum is described by Newton's equation along the tangential direction of the circular motion of the mass,

$$ma_t = -mg\sin\theta + f_d + f_r,$$

where $a_t = ld^2\theta/dt^2$. If the driving force is periodic as $f_d(t) = f_0\cos\omega_0 t$ and $f_r = -\kappa v = -\kappa l d\theta/dt$ then the equation of motion becomes

$$l\frac{d^2\theta}{dt^2} = -mg\sin\theta - \kappa l\frac{d\theta}{dt} + f_0\cos\omega_0 t. \tag{31}$$

If we rewrite Eq. (31) in a dimensionless form with $\sqrt{l/g}$ chosen as the unit of time, we obtain

$$\frac{d^2\theta}{dt^2} + q\frac{d\theta}{dt} + \sin\theta = b\cos\omega_0 t, \tag{32}$$

where $q = \kappa/m$, $b = f_0/ml$, and ω_0 is the angular frequency of the driving force. Solve Eq. (32) numerically and plot the trajectory in phase space when (1) $(\omega_0, q, b) = (2/3, 0.5.0.9)$ and (2)$(\omega_0, q, b) = (2/3, 0.5.1.15)$

Boundary-Value Problems

For a second-order differential equation,

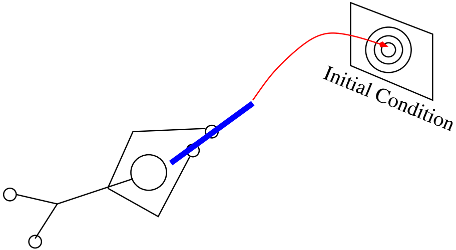
$$y'' = f(x, y, y'),$$

there are four possible boundary condition sets:

1. $y(x_0) = y_0$ and $y(x_1) = y_1$
2. $y(x_0) = y_0$ and $y'(x_1) = v_1$
3. $y'(x_0) = v_0$ and $y(x_1) = y_1$
4. $y'(x_0) = v_0$ and $y'(x_1) = v_1$

- Shooting Method
- Relaxation Method

Shooting Method



Basic idea: change the given boundary condition into the corresponding initial condition.

Prerequisite: **Secant Method** to find a root of equation and the basic **algorithm(s) for ODE**.

Shooting Method

Convert a single second-order differential equation

$$\frac{d^2y_1}{dx^2} = f(x, y_1, y_1')$$

into two first-order differential equations:

$$y_1' \equiv \frac{dy_1}{dx} = y_2$$

and

$$\frac{dy_2}{dx} = f(x, y_1, y_2)$$

with boundary condition, for example, $y_1(0) = u_0$ and $y_1(1) = u_1$.

Shooting Method

How to change the given boundary condition into the initial condition?

- $y_1(0)$ is given
- **guess** $y_1'(0) = y_2(0) \equiv \alpha$.
  - Here the parameter α will be adjusted to satisfy $y_1(1) = u_1$.
  - For this we will use the **secant method**.
- Let us define a function of α as

$$g(\alpha) \equiv u_\alpha(1) - u_1,$$

  - $u_\alpha(1)$ is the boundary condition obtained with the assumption that $y_2(0) = \alpha$
    - $u_\alpha(1)$ is calculated by the usual algorithm for initial value problem (for example, by applying Runge-Kutta method) with assumed initial value $y_2(0) = \alpha$.
  - u_1 is the true boundary condition.
- Using the secant method, find the value α which satisfy

$$g(\alpha) = 0$$

Example

Solve the differential equation,

$$\frac{d^2u}{dx^2} = -\frac{\pi^2}{4}(u+1)$$

with boundary condition,

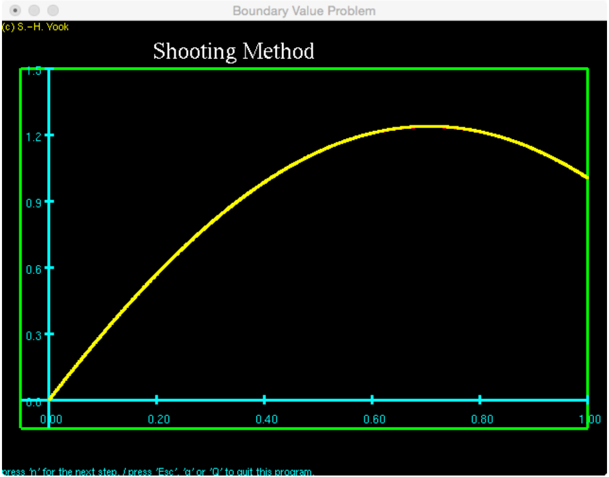
$$\begin{cases} u(0) = 0 \\ u(1) = 1 \end{cases}$$

Let

$$\begin{cases} \frac{dy_1}{dx} = y_2 \\ \frac{dy_2}{dx} = -\frac{\pi^2}{4}(u+1) \end{cases}$$

and assume that $y_1(0) = 0$ and $y_2(0) = \alpha$. Adjust α to satisfy

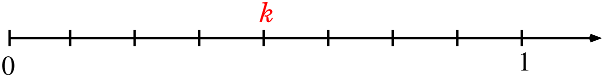
$$f(\alpha) = u_\alpha(1) - 1 = 0$$


Boundary Value Problem
(c) S.-H. Yook
Shooting Method
1.5
1.2
0.9
0.6
0.3
0.0
0.00
0.20
0.40
0.60
0.80
1.00
press 'n' for the next step. / press 'Esc', 'q' or 'Q' to quit this program.

Relaxation Method

$$\frac{d^2y}{dx^2} = f(x,y) \Rightarrow \frac{d^2y}{dx^2} - f(x,y) = 0 \tag{33}$$

(1) Divide the given interval into discrete N intervals (discretization).



(2) Use the definition of the numerical second order derivative to rewrite Eq. (33) as

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} - f(x_k, y_k) = 0 \tag{34}$$

at $x = x_k$. Eq. (34) becomes

$$y_{k+1} - 2y_k + y_{k-1} - h^2 f(x_k, y_k) = 0 \tag{35}$$

or equivalently

$$y_k = \frac{y_{k+1} + y_{k-1} - h^2 f(x_k, y_k)}{2} \tag{36}$$

Relaxation Method

(3) Using Eq. (36), keeping the boundary condition, iteratively calculate y_k as

$$y_k^{(n+1)} = \frac{y_{k+1}^{(n)} + y_{k-1}^{(n)} - h^2 f(x_k, y_k^{(n)})}{2}, \tag{37}$$

for all k. Here $y_k^{(n)}$ is the value of y_k at nth iteration.

Relaxation Method

Relaxation Method

$$y_k^{(n+1)} = \frac{y_{k+1}^{(n)} + y_{k-1}^{(n)} - h^2 f(x_k, y_k^{(n)})}{2}$$

Successive Over relaxation Method

$$y_k^{(n+1)} = w\left(\frac{y_{k+1}^{(n)} + y_{k-1}^{(n)} - h^2 f(x_k, y_k^{(n)})}{2}\right) + (1-w)y_k^{(n)}$$

where w is called as *over relaxation parameter* and $w \in [0, 2]$. Usually $w > 1$ is used to speed up the slow converging process and $w < 1$ is frequently used to establish convergence of diverging iterative process or speed up the convergence of an overshooting process.

Example: Relaxation Method

Solve the differential equation using the relaxation method,

$$\frac{d^2u}{dx^2} = -\frac{\pi^2}{4}(u+1)$$

with boundary condition,

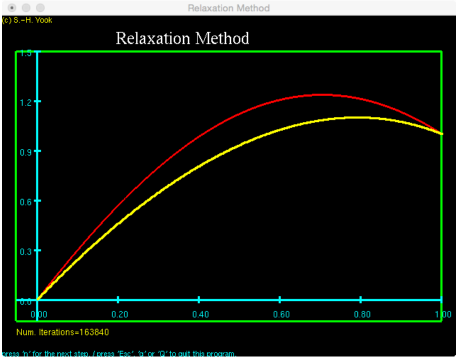
$$\begin{cases} u(0) = 0 \\ u(1) = 1 \end{cases}$$

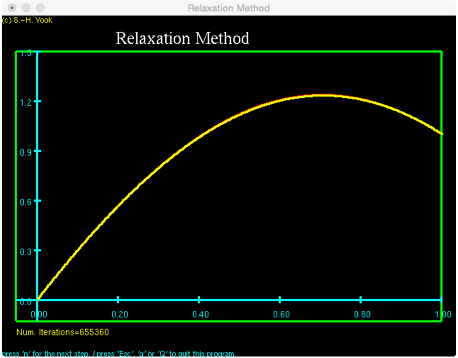
Let

$$\begin{cases} \frac{dy_1}{dx} = y_2 \\ \frac{dy_2}{dx} = -\frac{\pi^2}{4}(u+1) \end{cases}$$

and assume that $y_1(0) = 0$ and $y_2(0) = \alpha$.

Example: Relaxation Method





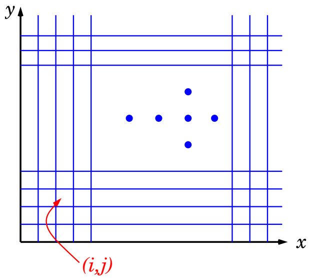
Relaxation Method in 2-Dimensional Space

Extending the relaxation method to 2-dimensional case is straightforward. Consider the partial differential equation:

$$\nabla^2 f = g(x, y) \tag{38}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = g(x, y) \tag{39}$$

Divide the given region into very small area.



Relaxation Method in 2-Dimensional Space

$$\frac{f(i+1,j)-2f(i,j)+f(i-1,j)}{h^2}+\frac{f(i,j+1)-2f(i,j)+f(i,j-1)}{h^2}=g(i,j)$$

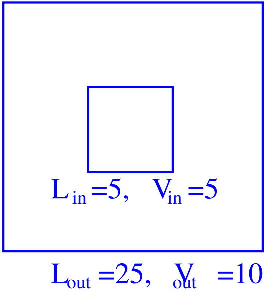
Relaxation Method in 2D

$$f^{(n+1)}(i,j)=\frac{f^{(n)}(i+1,j)+f^{(n)}(i-1,j)+f^{(n)}(i,j+1)+f^{(n)}(i,j-1)-h^2g(i,j)}{4}$$

Successive Over Relaxation Method in 2D

$$f^{(n+1)}(i,j)=w\,\frac{f^{(n)}(i+1,j)+f^{(n)}(i-1,j)+f^{(n)}(i,j+1)+f^{(n)}(i,j-1)-h^2g(i,j)}{4}+(1-w)f^{(n)}(i,j)$$

Homework



Use a relaxation method to compute the potential distribution between the two concentric square cylinders shown in the figure. The potential and the length of the square are given in the figure. Sketch the equipotential surface.