Roots of an equation
How to use your computer to find roots?

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Bisection Method

- The simplest and the most intuitive method.
- Find roots of equation $f(x) = 0$.
- Assume that we know the root $x_r$ is in the interval $[a, b]$.

![Bisection Method Diagram]

**Algorithm**

1. set $x_0 = \frac{a+b}{2}$
2. If $f(a)f(x_0) < 0$ then let $b = x_0$
3. else set $a = x_0$.
4. repeat 1-3 until $|a - b| \leq \delta$.

$\delta$ is a tolerance.

**Example**

Let $f(x) = e^x \ln x - x^2$ and find the root of an equation $f(x) = 0$. Hint:

$x = 1 \rightarrow f(1) = -1$ and $x = 2 \rightarrow f(2) = 2$ so let $a = 1$ and $b = 2$. 
Design for Visualization

\[ f(x) = \sin x + \ldots \]

results: \( x = 0.76423 \)
Find the solution of the equation

\[ f(x) = x^5 - 3x^4 - 5x^3 + x^2 + x + 3 = 0 \]

in the interval \( x \in [-2, 4] \) using bisection method.
Newton-Raphson Method

Assume a smooth function around its root.

\[ f(x_r) = 0, \quad x_r = \text{root} \]

Use Taylor expansion around \( x_r \).

\[
 f(x) = f(x_r) + (x_r - x)f'(x_r) + \cdots = 0 \tag{1}
\]

Idea: Let \( x_k \) be a trial value for the root of \( f(x) = 0 \) (i.e., \( x_r \)) at \( k \)-th step and approximate \( x_r \) at \((k + 1)\)-th step based on \( x_k \).

From Eq. (1)

\[
 f(x_{k+1}) \simeq f(x_k) + (x_{k+1} - x_k)f'(x_k) \simeq 0 \tag{2}
\]

\[
 x_{k+1} = x_k - \frac{f_k}{f'_k} = x_k + \Delta x_k \tag{3}
\]

Here \( f_k = f(x_k) \) and \( \Delta x_k \) is a kind of a correction.
Newton-Raphson Method

Iteration:

\[ x_{k+1} = x_k - \frac{f_k}{f'_k} = x_k + \Delta x_k \]

\[ \Delta x = -\frac{f(x_k)}{f'(x_k)} \]

- \( f'(x) > 0, \ f(x) > 0 \)
- \( f'(x) > 0, \ f(x) < 0 \)
Find the solution of the equation

\[ f(x) = x^5 - 3x^4 - 5x^3 + x^2 + x + 3 = 0 \]

in the interval \( x \in [-2, 4] \) using Newton-Raphson method.
Secant Method

- More generalized version of Newton-Raphson method.
- **Important:** This will be also used in shooting method to find a solution of differential equation with given boundary condition.
- If $f(x)$ has an implicit dependence on $x$.
  - Or if $f(x)$ is give by the numerical data (numbers).
  - Thus it is difficult to find out the derivative, $f'(x)$.

⇒ use the two points definition of $f'(x)$

From Eq. (3)

$$x_{k+1} \approx x_k - (x_k - x_{k-1}) \frac{f_k}{f_k - f_{k-1}} = x_i + \Delta x_i$$
Find the solution of the equation

\[ f(x) = x^5 - 3x^4 - 5x^3 + x^2 + x + 3 = 0 \]

in the interval \( x \in [-2, 4] \) using secant method.