Roots of an equation

How to use your computer to find roots?

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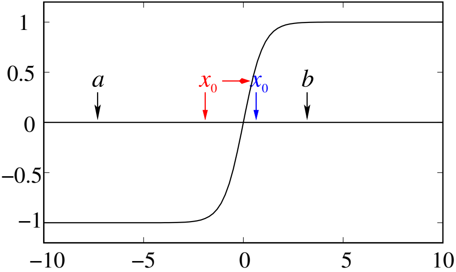
Bisection Method

- The simplest and the most intuitive method.
- Find roots of equation $f(x) = 0$.
- Assume that we know the root x_r is in the interval $[a, b]$.



Algorithm

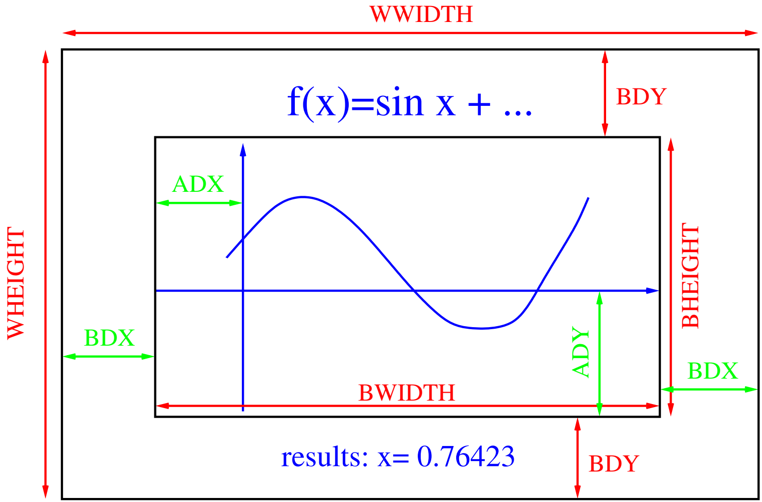
1. set $x_0 = \frac{a+b}{2}$
2. If $f(a)f(x_0) < 0$ then let $b = x_0$
3. else set $a = x_0$.
4. repeat 1-3 until $|a - b| \leq \delta$.

δ is a tolerance.

Example

Let $f(x) = e^x \ln x - x^2$ and find the root of an equation $f(x) = 0$. Hint: $x = 1 \rightarrow f(1) = -1$ and $x = 2 \rightarrow f(2) = 2$ so let $a = 1$ and $b = 2$.

Design for Visualization



Homework

Find the solution of the equation

$$f(x) = x^5 - 3x^4 - 5x^3 + x^2 + x + 3 = 0$$

in the interval $x \in [-2, 4]$ using bisection method.

Newton-Raphson Method

Assume a smooth function around its root.

$$f(x_r) = 0, \qquad x_r = \text{root}$$

Use Taylor expansion around x_r.

$$f(x) = f(x_r) + (x_r - x)f'(x_r) + \cdots = 0 \tag{1}$$

Idea: Let x_k be a trial value for the root of $f(x) = 0$ (i.e., x_r) at k-th step and approximate x_r at $(k+1)$-th step based on x_k.
From Eq. (1)

$$f(x_{k+1}) \simeq f(x_k) + (x_{k+1} - x_k)f'(x_k) \simeq 0 \tag{2}$$

$$x_{k+1} = x_k - \frac{f_k}{f'_k} = x_k + \Delta x_k \tag{3}$$

Here $f_k = f(x_k)$ and Δx_k is a kind of a correction.

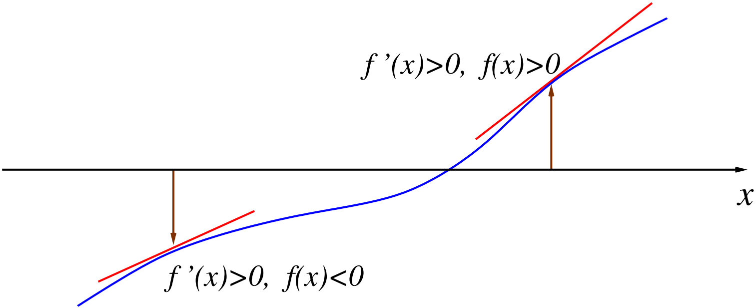
Newton-Raphson Method

Newton-Raphson Method

- Iteration:

$$x_{k+1} = x_k - \frac{f_k}{f'_k} = x_k + \Delta x_k$$

- $\Delta x = -\frac{f(x_k)}{f'(x_k)}$



Homework

Find the solution of the equation

$$f(x) = x^5 - 3x^4 - 5x^3 + x^2 + x + 3 = 0$$

in the interval $x \in [-2, 4]$ using Newton-Raphson method.

Secant Method

- More generalized version of Newton-Raphson method.
- Important: This will be also used in shooting method to find a solution of differential equation with given boundary condition.
- If $f(x)$ has an implicit dependence on x.
  - Or if $f(x)$ is give by the numerical data (numbers).
  - Thus it is difficult to find out the derivative, $f'(x)$.

$\Rightarrow$ use the two points definition of $f'(x)$

From Eq. (3)

$$x_{k+1} \simeq x_k - (x_k - x_{k-1})\frac{f_k}{f_k - f_{k-1}} = x_i + \Delta x_i$$

Homework

Find the solution of the equation

$$f(x) = x^5 - 3x^4 - 5x^3 + x^2 + x + 3 = 0$$

in the interval $x \in [-2, 4]$ using secant method.