Chapter 4. RWs on Fractals and Networks.

(Main Refs. [5], [9], [13], [*])

§1. RWs on Deterministic Fractals.

§2. Linear Excitation on Disordered lattice; Fracton; Spectral dimension d_s

§3. RWs on disordered lattice

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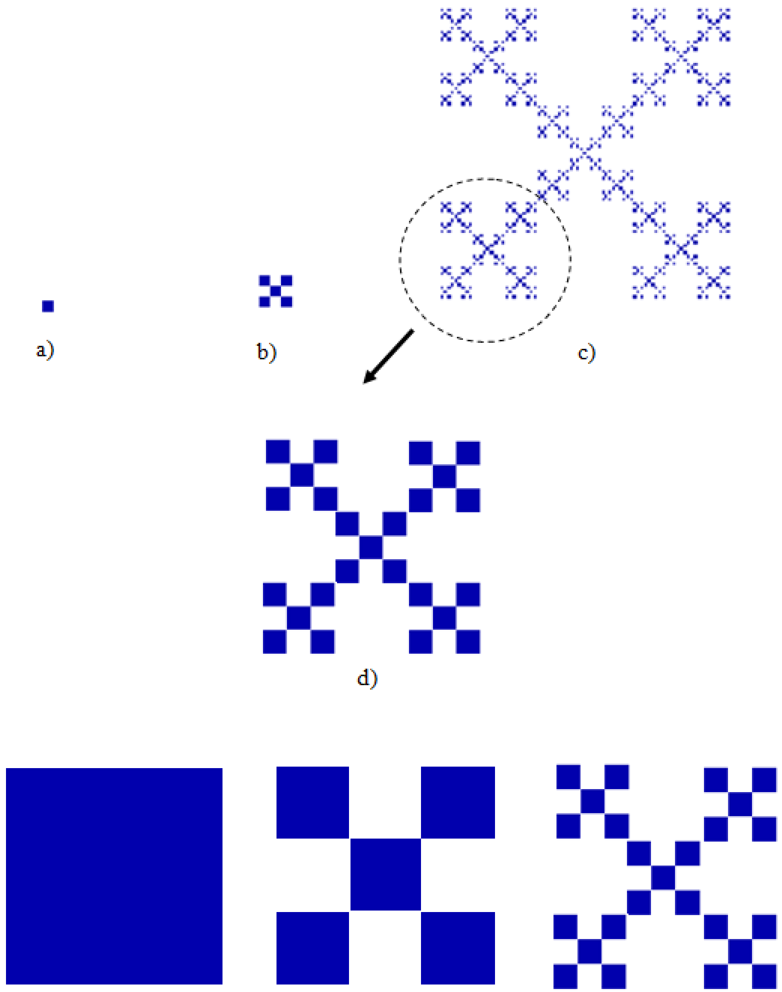
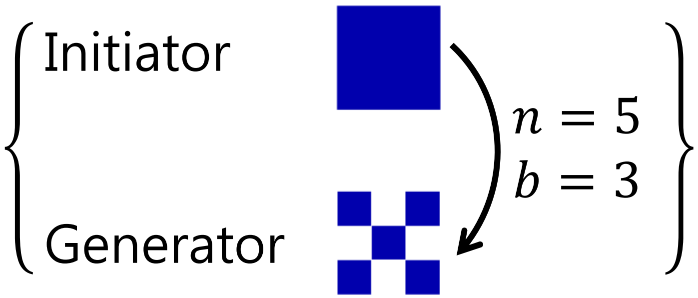
§1. RWs on Deterministic Fractals

① introduction to deterministic fractals





$\longrightarrow$ k^{th} generation prefractal

$$k \to \infty,\ M(L) = 5^k = L^{d_f} = \left(3^k\right)^{d_f}$$

$$d_f = \ln 5 \,/\, \ln 3 = \ln n \,/\, \ln b_0$$

(fractal dimension)

$$d_f = \ln n \,/\, \ln b_0 \qquad \textbf{(1)}$$

(b_0 is a special integer)

Self-similarity

$$\left(\begin{array}{l} M(R') = A(b)M(R) \quad \big(R \rightarrow R' = bR\ (b > 1)\big) \\ \\ \qquad A(b) = b^{d_f} \\ \\ M(R) = CR^{d_f} \end{array}\right. \qquad \begin{array}{r} \textbf{(2)} \\ \\ \\ \\ \textbf{(3)} \end{array}$$

Fractal self-similarity

$$M(R') = M(b_0^n R) = A(b_0^n)M(R)$$

(b_0 is a specific integer)

$$M(R) = f(R)R^{d_f}$$

$$M(b_0^n R) = f(b_0^n R)b_0^{nd_f} R^{d_f} = A(b_0^n)f(R)R^{d_f}$$

$$\left.\begin{array}{l} A(b_0^n) = b_0^{nd_f} \\ f(R) = f(b_0^n R) \end{array}\right\} \qquad \textbf{(4)}$$

$y = \ln R$

$f(\ln R) = \tilde{f}(y)$

$\tilde{f}(y) = \tilde{f}(y + n \ln b_0)$ (periodic function with the period $\ln b_0$) **(5)**

$M(R) = \tilde{f}(\ln R) R^{d_f} = \cos(\varphi \ln R) R^{d_f}$

$= \mathrm{Re}\left(R^{d_f + i\varphi}\right)$ **(6)**

$= \mathrm{Re}\left(R^{\tilde{\nu}}\right)$ $\left(\tilde{\nu} = d_f + i\varphi\right.$: complex fractal dimension$\left.\right)$

$(\varphi = 2\pi / \ln b_0)$

Generally $M(R) = \sum_{m=0}^{\infty} A(m) R^{d_f + im\varphi}$ **(7)**

$(\varphi = 2\pi / \ln b_0)$

② RWs on deterministic fractals

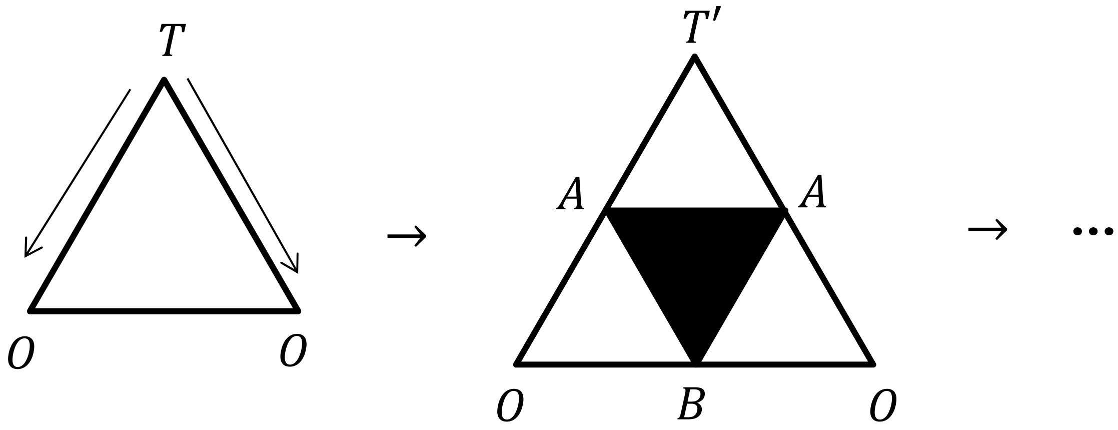
$$R_E^2 = \langle R^2 \rangle = N^{\frac{2}{dw}} \qquad (d_w > 2)$$

$$\left(\nu < \frac{1}{2} : \text{subdiff}\right)$$

Sierpinski Gasket $\quad (b_0 = 2, n = 3, d_f = \ln 3 \,/\, \ln 2)$

k=0 k=1 k=4

Renormalization of transit time ($T \to 0$) of RW



$$\left.\begin{aligned} T' &= T + A \\ 4A &= 4T + A + B + T' \\ 4B &= 4T + 2A \end{aligned}\right\} \longrightarrow \begin{aligned} T' &= 5T \\ A &= 4T \\ B &= 3T \end{aligned} \qquad \textbf{(8)}$$

$$\frac{1}{\nu} = d_w = \ln 5 \,/\, \ln b_0 = \frac{\ln 5}{\ln 2} \quad (\because\, b_0 \sim T^{\nu}) \qquad \textbf{(9)}$$

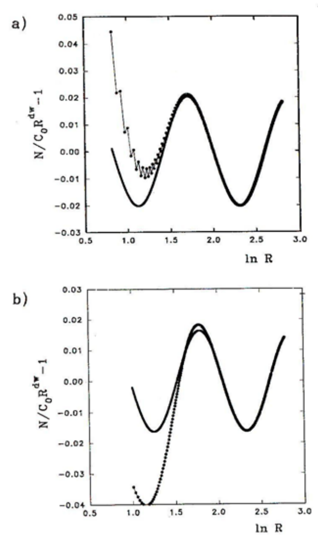
③ Simulation analysis of RWs on deterministic fractals

$$N = CR_E^{d_w}\left[1 + \sum_{m=1}^{\infty} C_m \cos(m\varphi \ln R + \delta_m)\right] \quad \text{(see Eq.(7))} \qquad \textbf{(10)}$$

(Introducing complex fractal dimension on d_w)

On checker board fractal

$$d_w = \ln 15 \,/\, \ln 3 = 2.46 \qquad \textbf{(11)}$$



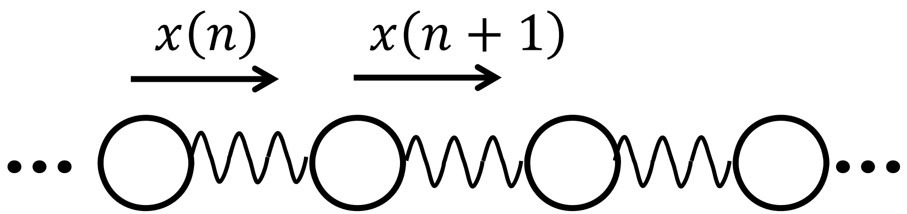
(Exercise) Prove $d_w = \ln 15 \,/\, \ln 3$ on checker board fractal.

§2. Linear Excitation on Disordered lattice; Fracton; Spectral dimension d_s

<u>1d lattice vibration</u>

$$\frac{d^2x(n,t)}{dt^2} = {\omega_o}^2[x(n+1,t) - 2x(n,t) + x(n-1,t)] \qquad \textbf{(11)}$$

$$(\omega_o^2 = k/m)$$



$$x(n+1,t) = \mathrm{Re}\ \phi(n)e^{i\omega t}$$

$$\phi(n+1) - 2\phi(n) + \phi(n-1) = -\lambda\phi(n) \qquad (\lambda = \omega^2/\omega_0^2) \qquad \textbf{(12)}$$

$$(\overleftrightarrow{\mathcal{L}}\vec{\phi} = -\lambda\vec{\phi})$$

1d RW master equation

$$\frac{dP(n,t)}{dt} = \frac{1}{2\tau}[P(n+1,t) - 2P(n,t) + P(n-1,t)] \qquad \textbf{(13)}$$

$$P(n,t) = \phi(n)\exp\left(-\frac{\lambda}{2\tau}t\right) \qquad \textbf{(14)}$$

$\longrightarrow$ the same eigenvalue equation with Eq. **(12)**

Lattice vibration on a cluster (fracton)

$$\frac{d^2 u(i,t)}{dt^2} = \omega_0^2 \left[\sum_{j(\mathrm{nn})}^{z_i} u(j,t) - z_i u(i,t) \right] \tag{15}$$

z_i is the coordination number of the site i in the cluster.
($z_i \leq z$: z is the coordination number of the base lattice.)

$$u_i(t) = \phi(i) \exp(-i\omega t) \tag{16}$$

$$\sum_{j(\mathrm{nn})} \phi_j - z_i \phi(i) = -\lambda \phi(i) \qquad (\lambda = \omega^2/\omega_0^2)$$

$$\left(\overleftrightarrow{\mathcal{L}_C} \vec{\phi} = -\lambda \vec{\phi} \right)$$

Density of states $\bar{\rho}(\lambda)$ or $\rho(\omega)$ in Eq **(2)** on nonrandom d-dimension lattices.

$$\rho(\omega) = \omega^{d-1} \quad (\omega \to 0) \qquad \textbf{(17)}$$

(or $\bar{\rho}(\lambda) = \lambda^{\frac{d}{2}-1}$)

Likewise on the cluster

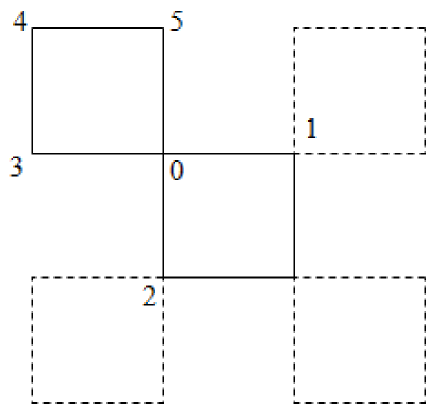
$$\rho(\omega) \approx \omega^{d_s-1} \quad (\omega \to 0) \qquad \textbf{(18)}$$

$d_s \to$ spectral dimension of the cluster (or fractal) **(19)**

§3. RWs on disordered lattice

"de Gennes → RW on disordered structure → Ant in labyrinth"

Hopping probability. $p(i,j) = p(i \to j)$



	$P(0,1)$	$P(0,2)$	$P(0,3)$	$P(0,5)$	$P(5,0)$	$P(5,4)$	$P(5,5)$
a)	1/4	1/4	1/4	1/4	1/4	1/4	1/2
b)	1/4	1/4	1/4	1/4	1/2	1/2	0
c)	1/4	1/4	1/4	1/4	1/4	1/4	0

Blind ant (a) :

$$\begin{cases} p(i,j) = \frac{1}{z} : \text{if } \langle ij \rangle \text{ is connected.} \\ p(i,i) = \dfrac{1 - z_i}{z} \end{cases} \quad \textbf{(20)}$$

Myopic ant (b) :

$$\begin{cases} p(i,j) = \frac{1}{z_i} : \text{if } \langle ij \rangle \text{ is connected.} \\ p(i,i) = 0 \end{cases} \quad \textbf{(21)}$$

Trapped ant (c):

$$\begin{cases} p(i,j) = \frac{1}{z} : \text{if } \langle ij \rangle \text{ is connected.} \\ \text{RW disappears with } p(i,j) = 1/z : \text{if } \langle ij \rangle \text{ is disconnected.} \end{cases} \quad \textbf{(22)}$$

→ (RW on the cluster disappears eventually.)

Blind ant

$$\frac{dP(i,t)}{dt} = \frac{1}{z\tau}\left(\sum_{j(\mathrm{nn})}^{z_i} P(i,t) - z_i P(i,t)\right) \qquad \textbf{(23)}$$

$$P(i.t) = \phi(i)\exp(-\bar{\lambda}t)$$

$$\sum_{j(\mathrm{nn})} \phi(j) - z_i\phi(i) = -\lambda \quad (\lambda = z\tau\lambda) \qquad \textbf{(16)}$$

$$\left(\rightarrow \overleftrightarrow{\mathcal{L}_C}\vec{\phi} = -\lambda\vec{\phi}\right)$$

(cf) Myopic ant

$$\frac{dP(i,t)}{dt} = \frac{1}{z_i\tau}\left(\sum_{j(\mathrm{nn})}^{z_i} P(j,t) - z_i P(i,t)\right)$$

$$\left(\rightarrow \overleftrightarrow{\mathcal{L}_G}\overleftrightarrow{D}^{-1}\vec{\phi} = -\lambda'\overleftrightarrow{\phi}\,(1-46)\right)$$

(Believe that myopic have the same critical property as blind ant.)

Lattice vibration

$$u(i,t) = u_0(q)\mathrm{Re}\sum_k \phi_k^*(q)\phi_k(i)\exp(-i\omega_k t)\,\big(u(i,0) = u_0(q)\delta_{iq}\big) \qquad \textbf{(24)}$$

Blind ant

$$P(i,t) = \mathrm{Re}\sum_k \phi_k^*(q)\phi_k(i)\exp\left(-\frac{\lambda_k}{z\tau}t\right) \qquad \big(P(i,0) = \delta_{iq}\big) \qquad \textbf{(25)}$$

$$P(R,t) = \frac{1}{N}\sum_{q=1}^{N}\frac{1}{N_R}\sum_{|1-q|=R}\mathrm{Re}\phi_k^*(q)\phi_k(i)\exp\left(-\frac{\lambda_k}{z\tau}t\right) \qquad \textbf{(26)}$$

$$P(0,t) = \frac{1}{N}\sum_k \exp\left(-\frac{\lambda_k}{z\tau}t\right) \qquad \textbf{(27)}$$

$$P(0,t) = \int_0^\infty d\lambda\bar{\rho}(\lambda)\exp\left(-\frac{\lambda}{z\tau}t\right) \qquad \textbf{(28)}$$

$$\lambda \leftrightarrow \omega^2$$

$$P(0,t) = \int_0^\infty d\omega\rho(\omega)\exp\left(-\frac{\omega^2}{z\tau}t\right) \qquad (\rho(\omega) = 2\bar{\rho}(\omega^2)\omega) \tag{29}$$

$$R_E^2 = \langle R^2 \rangle = t^{\frac{2}{d_w}} \ (= t^{2\nu_w})$$

$$P(0,t) = \frac{1}{R_E^{d_f}} = t^{-\frac{d_f}{d_w}} \tag{30}$$

$$\begin{aligned} P(0,t) &= \int_0^\infty d\omega\, \omega^{d_s-1}\exp\left[-\frac{\omega^2}{z\tau}t\right] \\ &= t^{-\frac{d_s}{2}}\int_0^\infty x^{\frac{d_s}{2}-1}\exp\left[-\frac{x}{z\tau}\right]dx \\ &\approx t^{-\frac{d_s}{2}} \end{aligned} \tag{31}$$

$$\underline{d_s = 2d_f/d_w} \tag{32}$$

$* \quad P(0,t) \approx \frac{1}{R_E^{d_f}} \approx \frac{1}{\mathfrak{D}(t)} \quad (?)$

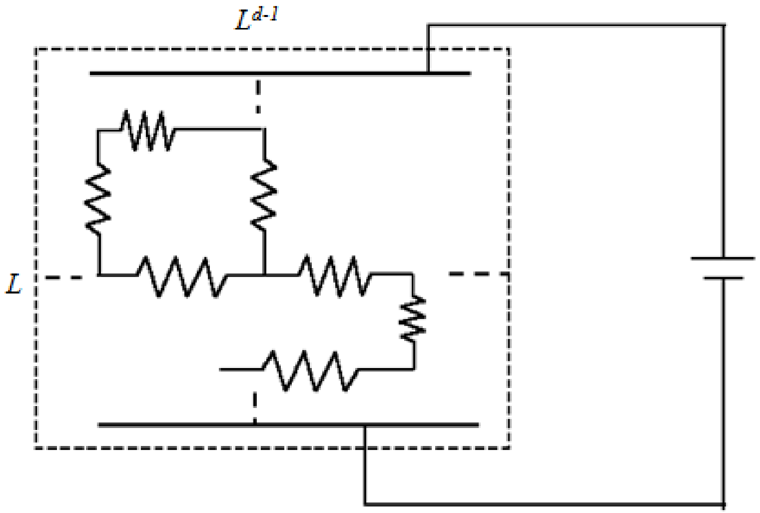
$\mathfrak{D}(t) \sim t^{\frac{d_s}{2}}$ (# of distinct visited sites)

(* Spectral dimension)

* On nonrandom lattice,

$d_w = 2 \left(\because d_f = d_s = d \right)$ **(33)**

§4. Random Resistor Network



$R(L)$: Resistance
$\sigma(L)$: Conductivity

(bond ↔ const. resistance)

$$R(L) = \frac{1}{\sigma(L)}\frac{L}{A} = \frac{1}{\sigma(L)}\frac{L}{L^{d-1}} = \frac{1}{\sigma(L)}L^{2-d} \quad \textbf{(34)}$$

$$R(L) = L^{\bar{\xi}}$$

(35)

$$\sigma(L) = L^{-\bar{\mu}}$$

($\bar{\xi}, \bar{\mu}$: finite-size exponents)

$$\bar{\xi} = 2 - d + \bar{\mu} \quad \textbf{(36)}$$

Einstein Relation

$$\sigma(L) = \frac{e^2}{k_B T} nD \sim L^{d_f - d + 2 - d_w} \sim L^{-\bar{\mu}} \quad \textbf{(37)}$$

$$\left(n = \frac{L^{d_f}}{L^d}, \qquad D = \frac{\langle R^2 \rangle}{2\mathrm{t}} \approx L^2 L^{-d_w} \right)$$

$$d_w = d_f - d + 2 + \bar{\mu} = d_f + \bar{\xi} \quad \textbf{(38)}$$

§5. RWs on Critical Percolation Clusters (CPCs)

* Cluster distribution function

$$\sum_s n_s s + P_\infty = p \quad \text{(Site percolation)} \tag{39}$$

$$\begin{cases} P_\infty & = \text{Prob. of the infinite cluster} \\ n_s s & = s\text{-sized cluster distribution function} \end{cases}$$

* Average $\begin{cases} \text{Largest-cluster (LC) average} \\ \text{All-cluster (AC) average} \end{cases}$

RWs on LC at $p = p_c$

$$\begin{cases} \text{LC} \rightarrow s\text{-sized cluster} \\ R_s \text{ : Radius of } s\text{-cluster, } d_p\text{: fractal dimension of LC} \end{cases}$$

$$R_{Es}^2 = \begin{cases} t^{\frac{2}{d_w}} & : R_{Es}^2 < R_s^2 \left(s^{2/d_p}\right) \\ R_s^2 & : R_{Es}^2 > R_s^2 \end{cases} \quad \textbf{(40)}$$

RWs on ACs at $p = p_c$ $\qquad [R_E^2] \sim t^{\frac{2}{d_w'}}$

$$[R_E^2] = \sum_s^{\infty} R_{Es}^2\, sn(s,p) = \sum_{s=0}^{s_c} R_s^2 s^{1-\tau} + \sum_{s_c}^{\infty} t^{2/d_w} s^{1-\tau} \quad (R_{sc} \sim s_c^{\frac{1}{d_p}} \sim t^{\frac{1}{d_w}})$$

$$\approx \int_0^{s_c} R_s^2 s^{1-\tau} ds + s_c^{\frac{2}{d_p}} \int_{s_c}^{\infty} s^{1-\tau} ds \approx s_c^{\frac{2}{d_p}+2-\tau} \approx t^{2/d_w'} \quad \textbf{(41)}$$

From $\tau = 1 + d/d_p$ and $d_p = d - \beta/\nu$

$$d_w' = \frac{1}{\nu_w'} = d_w(1 - \beta/2\nu)\ (> d_w) \qquad (\nu_w' < \nu_w) \quad \textbf{(42)}$$

RWs on LC and AC at $p \to p_c^-$ and $p \to p_c^+$

* for $\xi_p^2 < R_E^2$,

$$R_E^2 \sim t^{2/d_w}$$

$$[R_E^2] \sim t^{2/d_w'} \tag{43}$$

regardless of $p \to p_c^-$ and $p \to p_c^+$

* for $\xi_p^2 > R_E^2$ and $p \to p_c^-$

Infinite cluster × ($P_\infty = 0$)

$$\rightarrow \quad \begin{array}{l} R_E^2 \sim s^{2/d_p} \\ [R_E^2] \approx [s^{2d_p}] = (p_c - p)^{-2\nu+\beta} \end{array} \tag{44}$$

* for $\xi_p^2 > R_E^2$ and $p \to p_c^+$

Infinite cluster O ($P_\infty > 0$). $\to$ $\begin{cases} \text{homogeneous cluster} \\ \text{Euclidean space (not fractal)} \end{cases}$

$$R_E^2 \approx D(p)t$$
$$[R_E^2] \approx [\mathrm{D(p)}]\mathrm{t} \tag{45}$$

$$\sigma(p) = \frac{e^2}{k_B T} n[D(p)] = \begin{cases} (p - p_c)^\mu & p > p_c \\ 0 & p < p_c \end{cases} \tag{46}$$

$$\sigma(L, p) \approx L^{-\bar{\mu}} f_\sigma(L/\xi_p)$$
$$\approx L^{-\bar{\mu}} \bar{f_\sigma}(L^{\frac{1}{\nu}}(p - p_c)) \to \bar{\mu} = \frac{\mu}{\nu} \tag{47}$$

$$[D(p)] = P_\infty D(p) \to D(p) = (p - p_c)^{\mu - \beta} \qquad \left(\because P_\infty = (p - p_c)^\beta\right) \tag{48}$$

$$R_E^2 \approx D(p)t \approx (p - p_c)^{\mu - \beta} t$$
$$[R_E^2] \approx [D(p)]t \approx (p - p_c)^\mu t \tag{49}$$

Crossover scaling of $[R_E^2]$

$$[R_E^2] = t^{2/d'_w} f_{RW}(t^x(p - p_c)) \quad \textbf{(50)}$$

$$\begin{cases} \text{i)}\ f(y=0) = \text{const} \\ \text{ii)}\ y = -\infty, \text{from } (\mathbf{44}) \left([R_E^2] \approx (p_c - p)^{\beta - 2\nu}\right) \\ \qquad x(\beta - 2\nu) + 2/d'_w = 0 \\ \text{iii)}\ y = \infty,\ \ \text{from } (49)\ ([R_E^2] \approx (p_c - p)^{\mu} t) \\ \qquad 2/d'_w + x\mu = 1 \end{cases}$$

$$1 - x\mu = x(2\nu - \beta) = \frac{2}{d'_w} \quad \textbf{(51)}$$

$$x = \frac{1}{2\nu - \beta + \mu} \quad \textbf{(52)}$$

$$d_w = \left(1 - \frac{\beta}{2\nu}\right) d'_w = 2 + \frac{\mu - \beta}{\nu} \quad \textbf{(53)}$$

Crossover time from fractal property to Euclidean (homogeneous) prop.

$$t_c^x(p - p_c) \approx 1 \tag{54}$$

$$t_c \approx \xi_p^2 / D(p) \approx \frac{\xi_p^2 P_\infty}{[D(p)]} = (p - p_c)^{\beta - 2\nu + \mu} \quad (\xi_p^2 \sim D(p) t_c)$$

$$P(0, t) \approx t^{-d_s/2} \quad \text{(LC)}$$

$$\left(d_s = 2d_p/d_w\right)$$

$$[P(0, t)] = t^{-d_s'/2} \quad \text{(AC)} \tag{55}$$

$$\left(d_s = 2d_p/d_w'\right)$$

Alexander-Orbach conjecture

Percolation (Geometrical) Exponents

$\beta, \nu, \gamma, \sigma, \tau$

$\Updownarrow$ Relation

Dynamic Exponents

$d_w(d'_w), \mu, d_s$

$$d_s = \frac{4}{3} \qquad \text{on CPC} \tag{56}$$

$$\left(d_w = \frac{8d_p}{3}\right) \qquad \text{of any } d \ (1 < d \leq 6)$$

No proof at all !!

§6. $P(r,t)$ on CPC & fractals

$$P(r,t) = \frac{1}{\left(t^{1/d_w}\right)^{d_f}} f\left(\frac{|\vec{r}|}{t^{1/d_w}}\right)$$

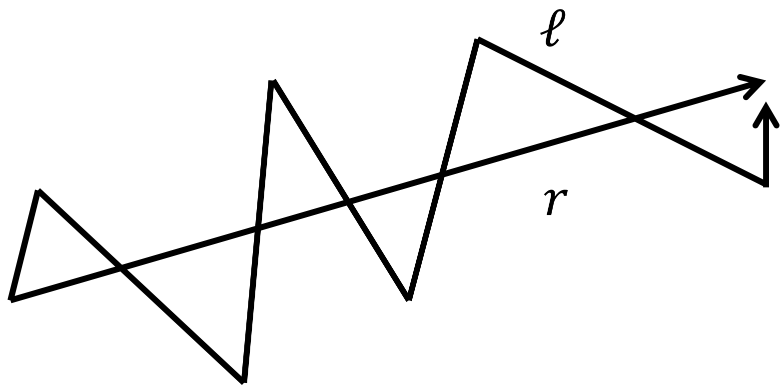
$$= t^{-d_s/2} f_p\left(r^{d_w}/t\right) \qquad \textbf{(57)}$$

$\underline{f_p(0) = \text{const.}}$

$$f_p(y) = \exp(-y^a) \qquad \textbf{(58)}$$

$$\begin{cases} a = 1 & \text{(conjecture 1)} \\ \dfrac{1}{d_w - 1} \le a \le \dfrac{d_{\min}}{d_w - d_{\min}} & \\ \left(a = \dfrac{d_{\min}}{d_w - d_{\min}}\right) & \text{(conjecture 2)} \\ a = \dfrac{1}{d_w - 1} & \text{(conjecture 3)} \end{cases} \qquad \textbf{(59)}$$

d_{min} is the fractal dimension of the chemical distance ℓ : $\ell \sim r^{d_{min}}$



Flory approximation for SAWs on CPCs

$$F = \frac{C_{re} N^2}{R^{d_f}} + k_B T \ln P(R, N)$$

$$\nu_{SAW} = \frac{2 + a}{d_f + a d_w} \quad \textbf{(60)}$$

§7. Renormalization group for d_s and d_w

* If one knows two among d_f, d_s, d_w and $\mu(\xi)$, one knows all.

RG trans,

$\lambda' = f_b(\lambda)$ (b : scaling factor) **(61)**

(fixed point $\lambda^* = 0$ → Laplacian property)

$$\lambda' = \frac{df_b}{d\lambda}\Big|_{\lambda=0} \lambda = b^{\theta}\lambda \qquad \textbf{(62)}$$

$$\theta = \ln \frac{df_b}{d\lambda}\Big|_{\lambda=0} / \ln b \qquad \textbf{(63)}$$

Integrated density of states

$$I(\lambda) = \int_0^{\lambda} \bar{\rho}(\lambda) d\lambda \approx \lambda^{\frac{d_s}{2}}$$

$$I(\lambda) = I(\lambda')b^{-d_f} = b^{-d_f}\left(b^{\theta}\lambda\right)^{\frac{d_s}{2}} \quad \textbf{(64)}$$

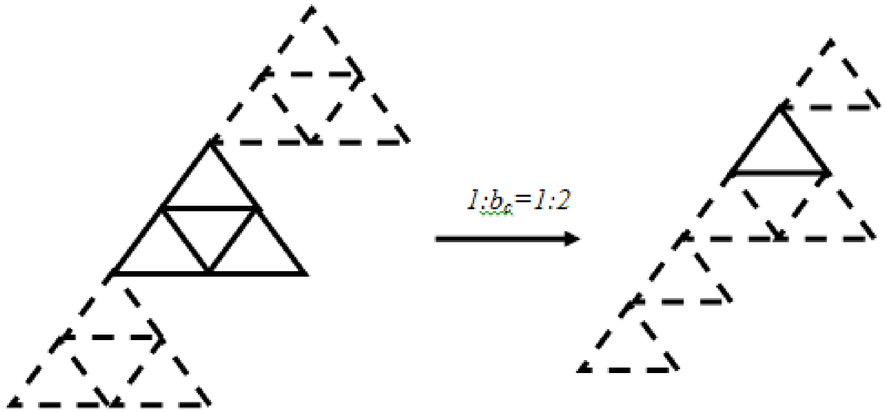
$$d_s = \frac{2d_f}{\theta} \quad \textbf{(65)}$$

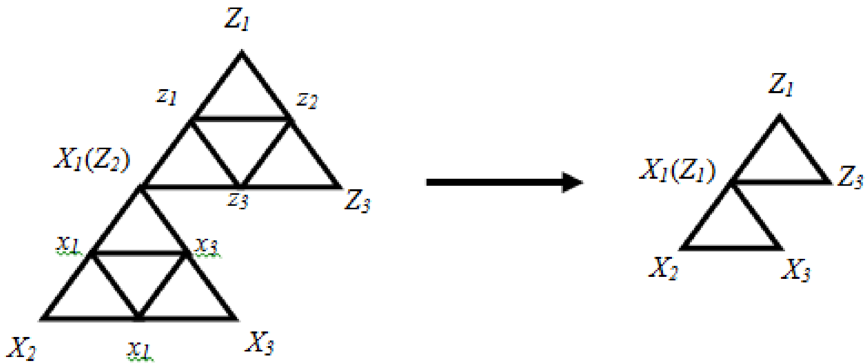
$$\theta = d_w \quad \textbf{(66)}$$

* RW $\rightarrow P(r,t) \sim \exp(-\lambda t)$

$$\left\{\begin{matrix} r \rightarrow 1\colon b \\ \lambda \rightarrow 1\colon b^{-\theta} \\ t \rightarrow 1\colon b^{+\theta} \end{matrix}\right\} \rightarrow \; r^{d_w} \sim t \; \rightarrow \theta = d_w$$

(Exercise) From the renormalization group scheme of the Sierpinski gasket as in the following Figure,





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① $\lambda' = f_b(\lambda) = f_v(\lambda) = 5\lambda - \lambda^2$

② $d_w = \ln 5\,/\,\ln 2\,, d_s = \ln 9\,/\,\ln 5$

§8. First Passage Time on the Network (or graphs)

$$P_{ij}(t) = \delta_{t0}\delta_{ij} + \sum_{t'=0}^{t} P_{jj}(t-t')F_{ij}(t') \qquad \left(P_{ij}(0) = \delta_{ij}\right) \tag{67}$$

$$\tilde{P}_{ij}(s) = \sum_{t=0}^{\infty} e^{-st}\, P_{ij}(t)\,, \qquad \tilde{F}_{ij}(s) = \sum_{t=0}^{\infty} e^{-st}\, F_{ij}(t)$$

$$\tilde{F}_{ij}(s) = \frac{\tilde{P}_{ij}(s) - \delta_{ij}}{\tilde{P}_{jj}(s)} \tag{68}$$

$$\langle T_{ij} \rangle = \sum_{t=0}^{\infty} t F_{ij}(t) = -\tilde{F}'_{ij}(0) \tag{69}$$

$$R_{ij}^{n} \equiv \sum_{t=0}^{\infty} t^{n} \{ P_{ij}(t) - P_j(t = \infty) \} \tag{70}$$

$$\{ P_{ij}(t) - P_j(t = \infty) \} \xrightarrow{t \to \infty} 0 \quad \text{and} \quad P_j(t \to \infty) = \frac{k_j}{N\langle k \rangle} \tag{1-(49)}$$

$$\tilde{P}_{ij}(s) = \frac{k_j}{(\sum_l k_l)(1 - e^{-s})} + \sum_{n=0}^{\infty} (-1)^n R_{ij}^{n} \frac{s^n}{n!} \tag{71}$$

$$\langle T_{ij} \rangle = \begin{cases} \dfrac{1}{P_i^{\infty}} & \text{for } (j = i) \\ \dfrac{\left[R_{jj}^{(0)} - R_{ij}^{(0)}\right]}{P_j^{\infty}} & \text{for } (j \neq i) \end{cases} \quad \textbf{(72)}$$

$$R_{jj}^{(0)} - R_{ij}^{(0)} = \sum_{t=0}^{\infty} \left[P_{jj}(t) - P_{ij}(t)\right]$$

§9. RWs and Structure of Complex Networks

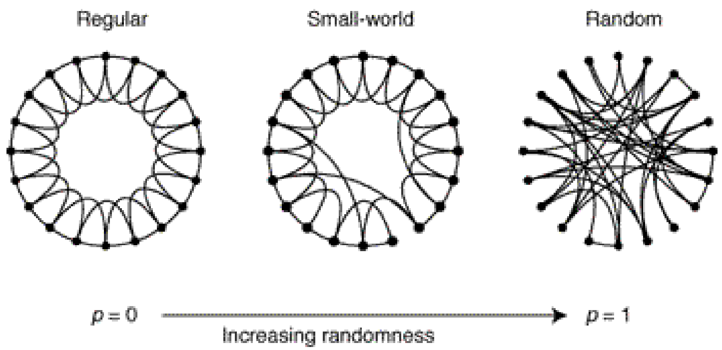
① Networks

* Random Networks

Connect each pair of nodes with given probability p

Degree distribution : $P(k) \sim \dfrac{e^{-\langle k \rangle}\langle k \rangle^{k}}{k!}$

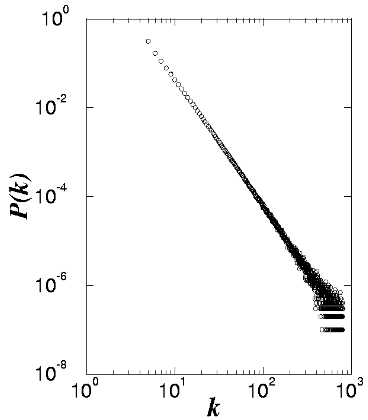
* Small-World (SW) Networks : Watts-Strogatz (WS) model



Interpolate between regular and random network

* Scale-free (SF) Networks

Degree distribution : $P(k) \sim k^{-\gamma}$



② Random walks on SW networks

There is a characteristic time scale : ξ^2

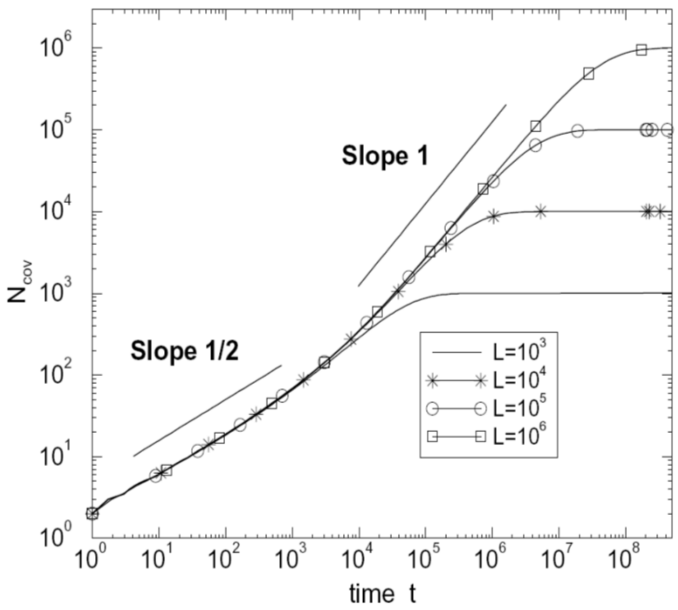
$\xi = 1/p$ (p : rewiring probability → related to shortcut density)

$t \ll \xi^2$ → the walker does not meet the shortcut → only see the regular structure

$$\mathcal{D}(t) \sim t^{\frac{1}{2}} \quad \textbf{(1-27)}$$

$t \gg \xi^2$ → the walker can move to new region of networks

$$\mathcal{D}(t) \sim t \quad \textbf{(1-27)}$$



Average number of distinctive visited nodes

③ Number of distinct visited sites : $\mathcal{D}(t)$

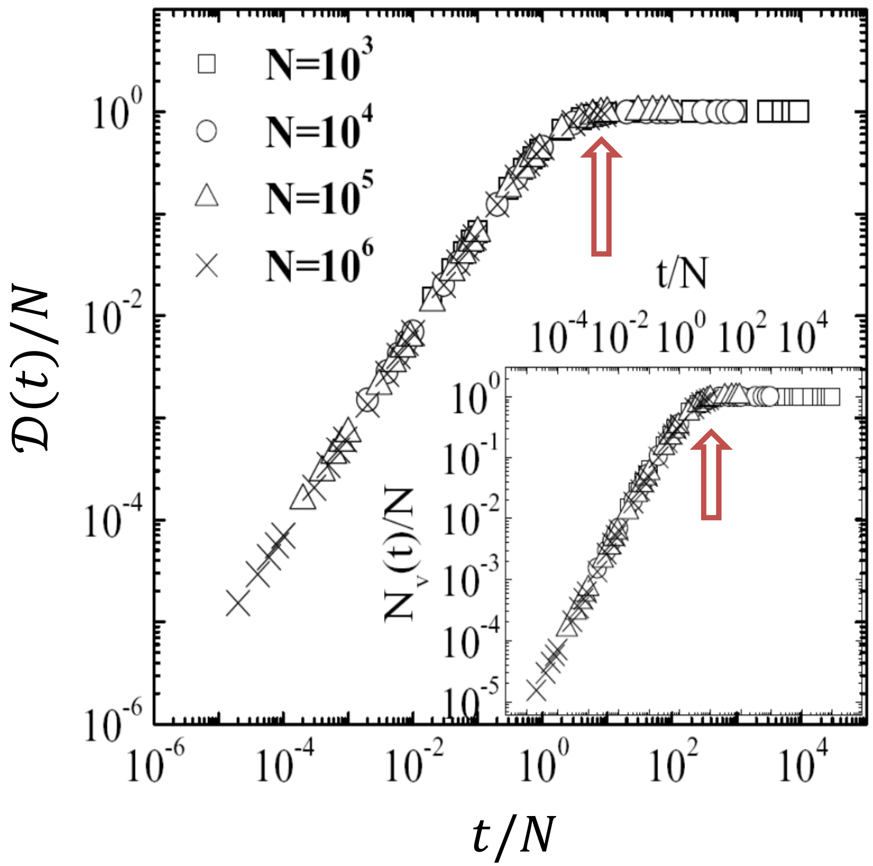
On SF networks,

$\mathcal{D}(t) = Nf(t/N)$

$f(x) \sim \begin{cases} x & ,x \ll 1 \\ 1 & ,x \gg 1 \end{cases}$

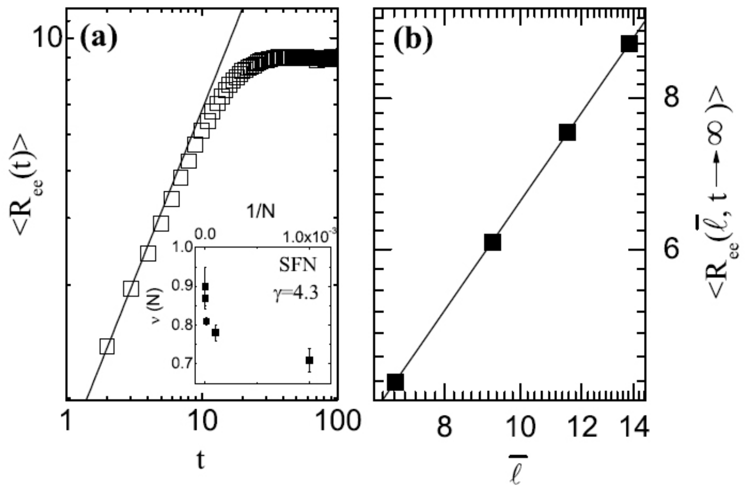
$t_\times \sim N$

at $t = t_\times$ $\mathcal{D}$ saturates to a constant value



④ Scaling relation for end-to end distance of RW on SF networks

$\gamma = 4.3$



$R_E \sim t^{\nu}$ $\quad (\nu = 1)$

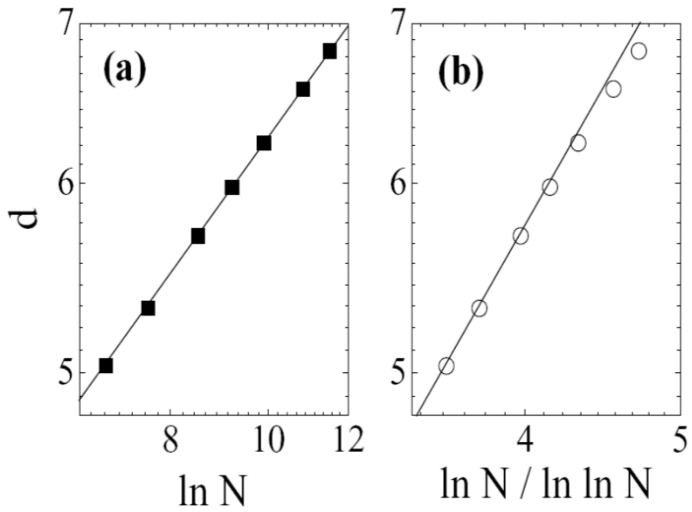
(From the results on a Caylay tree, we expect that $\nu = 1$)

$R_E(t \to \infty) \sim \bar{\ell}^{\alpha}$ $\quad (\alpha = 1)$

($\bar{\ell}$: the minimum distance averaged over all possible pairs of sites (or nodes))

$R_E(t)$ does not increase indefinitely, but reaches a saturation value $R_E(t \to \infty) \sim \bar{\ell}$ after a cross over time τ_E.

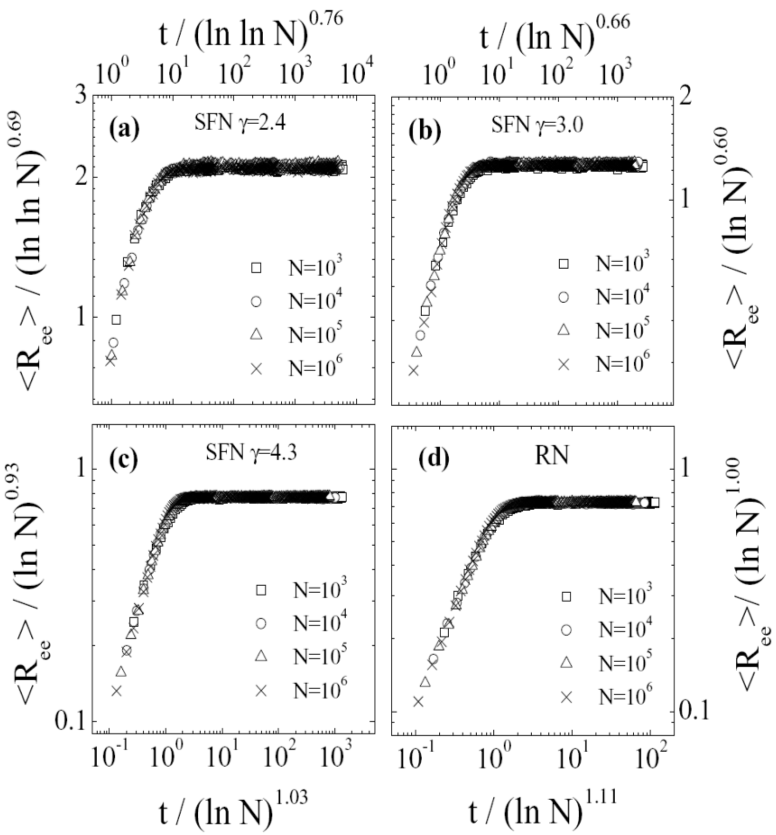
$$d(N) \sim \begin{cases} \ln\ln N & ,\ 2 < \gamma < 3 \\ \ln N / \ln\ln N & ,\ \gamma = 3 \\ \ln N & ,\ \gamma > 3 \end{cases}$$



$$\bar{\ell}(\gamma, N) \sim d \sim \begin{cases} \ln\ln N & ,2 < \gamma < 3 \\ \ln N & ,\gamma \geq 3 \end{cases}$$

⑤ Possible application of the scaling relation for R_E of RW on SF networks

z characterizes the time at which R_E saturate or when the walker feels the finite-size



$$R_{E(\ell)} = \bar{\ell}(\gamma, N) g(t/\bar{\ell}^z)$$

$$g(x) \sim \begin{cases} x^{\nu} & , x \ll 1 \\ \text{const.} & , x \gg 1. \end{cases}$$

$$\bar{\ell}(\gamma, N) \sim d \sim \begin{cases} \ln \ln N & , 2 < \gamma < 3 \\ \ln N & , \gamma \geq 3 \end{cases}$$

From this scaling relation we expect that the computing time needed for the measurement of scaling behavior of d by RW method increases as $O((\ln N^z))$ or $O((\ln \ln N)^z)$